A Comparative Study on the L_1 Optimal Event-Based Method for Biped Walking on Rough Terrains

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ABSTRACT This paper is concerned with a comparative study of biped walking on rough terrains. Given a bipedal robot capable of walking on a flat ground with periodic behavior, whose motion can be described by a limit cycle with the Poincaré map, we consider whether the robot remains stable on rough terrain, in which geometrical uncertainties of the terrain are assumed to be persistent and bounded. More precisely, the l_∞-induced norm is defined on the Poincaré map and taken as a performance measure evaluating a robot walking with the bounded persistent uncertainties. To minimize the performance measure and achieve an optimal walking performance, we further provide a systematic controller design scheme consisting of a inner-loop continuous-time controller and a outer-loop event-based controller, in which the latter is described as a sort of the l_1 optimal controller. Finally, the validity as well as the effectiveness of our proposed methods in biped walking on a rough terrain are demonstrated through simulation studies.

INDEX TERMS Bipedal robots, legged locomotion, robust uneven terrain walking, limit cycle, event-based control.

I. INTRODUCTION

There have been a number of studies on stable walking of bipedal robots in structured environments (such as labs and indoors), which have shown successful demonstrations through simulations and experiments [1]–[3]. Consequently, stable and robust walking of bipedal robots on unstructured terrains has recently attracted much attention of robotics community, and various control methods for biped walking have been developed in [4]–[9].

A. OBJECTIVES OF THIS PAPER

With respect to the aforementioned various methods, this paper is in a position to take advantage of existing available controllers which are capable of stable *level ground* walking, and aims at improving their robustness against external disturbances arising from uneven terrains. More precisely, this issue would be interpreted as a problem of robust performance analysis throughout the present paper, and the

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following two questions should be tackled to establish the corresponding problem formulation.

(a) How can we evaluate/compare the performance of biped walking on the uneven ground in the sense of disturbance rejection?

(b) If we are equipped with such a measure, can we utilize it to improve performance of a walking controller?

To put it another way, these two questions motivate us to construct a detailed robust control problem relevant to stable walking of bipedal robots on rough terrains, and it could be classified by the following sub-problems.

(A) **Sub-problem 1** Define an adequate measure. One could determine how well the biped walking robot keeps from falling in the presence of disturbances by defining an adequate performance measure.

(B) **Sub-problem 2** Design an optimal controller based on the proposed measure. If a solvable problem corresponding to an optimal controller synthesis could be formulated, robust outdoor walking of bipedal robots would be possible.

To resolve the sub-problems above, this paper takes a hierarchical feedback control architecture. In other words, the main concern of this paper is to design an *outer loop* controller via an event-based control scheme to achieve robustness and optimality of a periodic trajectory for rough terrains, in which the corresponding periodic motion is assumed to be constructed by an *inner loop* feedback controller based on the conventional limit cycle walking paradigm [10].

B. RELATED STUDIES ON POINCARÈ MAP

The advantages and expectations of our control architecture can be described as follows. First of all, the stability of a biped walking robot is expected to be verified by exploiting the fact that the orbital stability of a periodic trajectory has the one-to-one correspondence with the stability of an associated Poincarè map, as shown in the previous studies [10]–[12]. This architecture can also make direct applications of robust/ optimal control theories developed in linear time-invariant (LTI) systems to nonlinear complex biped walking systems to be simple, by defining an adequate performance measure on the linearized Poincarè map. For example, one of the most representative methods in linear control theories, so-called the linear quadratic regulator (LQR), has been used in [13], [14] for achieving stable walking of robot systems. Aiming at robust performance for unknown elements in biped walking systems, the methods of the H_2 optimal control and the H_∞ optimal control have been employed in [15] and [16], respectively, while a min-max problem is also solved in [17] to reduce the response of the systems for a sort of step-down which can be also mathematically represented by an impulse disturbance.

Here, it should be remarked that disturbances of practical importance occurring during outdoor walking cannot be limited to those of finite energy, but the aforementioned studies are confined themselves to dealing with such disturbances. Hence, it is quite important to develop a systematic method for tackling more realistic disturbances originated from rough terrains consisting of pebbles for example, which should be mathematically represented as bounded and persistent disturbances.

C. CONTRIBUTIONS AND ORGANIZATION OF THIS PAPER

With regard to robustness for bounded persistent disturbances, this paper is concerned with the treatment of the l_1 optimal control theory in walking systems. More precisely, the main idea of this paper is to take the l_∞-induced norm as a quantitative performance measure on the linearized Poincarè map relevant to biped walking systems for bounded persistent disturbances. The employment of the l_∞-induced norm is theoretically meaningful since this norm corresponds to the l_1 norm of the impulse response and it is essentially equivalent to the maximum magnitude of the response for the worst input with a unit amplitude (i.e., the worst bounded persistent input). At each step a bipedal robot interacts with the ground, we consider the synthesis of a sort of event-based controllers based on such a quantitative performance

measure. In other words, the l_1 optimal event-based controller adequately updates some parameters of the inner-loop controller, by which the proposed measure (i.e., the l_∞-induced norm) could be minimized. More importantly, it would be shown that the synthesis problem of the l_1 optimal event-based controller can be equivalently transformed to that of the (pure) discrete-time l_1 optimal control, for which there are some useful solution tools as discussed in [18], [19]. As this will suppress the worst possible behavior, the walking performance of a bipedal robot for persistent terrain disturbances could be improved considerably. Another advantage of the proposed framework is that it requires no particular form of the inner-loop closed system.

The contributions of this paper could be summarized as follows. This paper provides a systematic framework for improving robustness of biped walking systems against bounded persistent disturbances for the first time. This new framework would be also shown to be converted to a solvable problem of the so-called (pure) discrete-time l_1 optimal control. The characteristics of this framework in biped walking systems are also theoretically analyzed by comparing other Poincarè map-based frameworks. Simulation results with a nonlinear dynamic walking model on rough terrains that their slope or height are randomly varied are given to demonstrate the effectiveness of the framework. Finally, it should be remarked that the arguments introduced in this paper are significant extensions of the partial results presented at the conferences [20], [21], in which rigorous mathematical/ theoretical arguments were omitted.

The organization of this paper is as follows. The problem formulation associated with biped walking systems is provided in Section II. The main results of this paper relevant to the l_1 optimal event-based control are given in Section III, and their theoretical comparisons to other Poincarè map-based frameworks are discussed in Section IV. Simulation results with existing simple biped walking model are presented in Section V, and the application of the proposed method to failure prediction is thoughtfully discussed in Section VI. Finally, the concluding remarks are given in Section VII.

The notations used in this paper are as follows. The notations $\mathbb{N}$ and $\mathbb{R}^\nu$ imply the set of positive integers and the set of ν-dimensional real vectors, respectively, while the notation $\mathbb{N}_0$ implies $\mathbb{N} \cup \{0\}$. We use the notation $\| \cdot \|_\infty$ to mean the l_∞ norm of a vector sequence, i.e.,

$$\|g(\cdot)\|_\infty := \max_i \sup_{k \in \mathbb{N}_0} |g_i(k)|,$$

and the matrix ∞-norm, i.e.,

$$\|T\|_\infty := \max_i \sum_j |T_{ij}|,$$

and the l_∞-induced norm of an operator T, i.e.,

$$\|T\|_\infty := \sup_{\|w\|_\infty \neq 0} \frac{\|(Tw)(\cdot)\|_\infty}{\|w\|_\infty},$$

whose distinction will be clear from the context.

II. PROBLEM FORMULATION FOR BIPED WALKING

This section constructs the problem formulation for biped walking on rough terrains. To put it another way, this section introduces a Poincarè map-based method for analyzing biped walking motions through a limit cycle. This method is based on a hierarchical feedback control architecture consisting of an inner-loop controller and an outer-loop controller, in which the parameters of the former are updated by the latter. More importantly, it would be shown that this involved hierarchical control architecture could be converted to a standard robust/optimal control problem through a linearization of the corresponding Poincarè map.

A. BIPED WALKING MODEL BASED ON POINCARÉ MAP

This section considers the treatment of the walking motion as a limit cycle, similarly for the arguments in [12], [22]. The behavior of the limit cycle can be described by the first return map, i.e., the Poincarè map, which relates the states of a walking robot between successive steps. In other words, the Poincarè map $\mathbf{x} \mapsto \mathbf{P}(\mathbf{x})$ can be described by

$$\mathbf{x}_{k+1} = \mathbf{P}(\mathbf{x}_k) \qquad (1)$$

where $\mathbf{x}_k = \mathbf{x}(t_k) \in \mathbb{R}^n$ represents the n-dimensional state of the robot at k-th step, i.e., k-th intersection between the robot trajectory and the Poincarè section with time $t = t_k (k \in \mathbb{N}_0)$.

Next, an inner-loop controller of a bipedal robot is supposed to be represented by

$$\boldsymbol{\tau}(t) = \Gamma(\mathbf{x}(t), \mathbf{u}) \qquad (2)$$

(i.e., the parameterized (continuous-time) state-feedback form), where $\mathbf{u} \in \mathbb{R}^{n_u}$ is the parameter vector of the controller and $\boldsymbol{\tau}(t) \in \mathbb{R}^{n_\tau}$ is the input vector at time t. We also assume additional constraint to properly formulate the problem such that the adjustable control parameter vector $\mathbf{u}$ is updated in a discrete-time manner only when the system trajectory jumps on the Poincarè section and remains the same during its continuous-time dynamics.

Consequently, we define the n_w-dimensional unknown disturbance vector $\mathbf{w}$ originating from uneven terrain. The type of disturbances that we deal with in this paper only include the effect of ground geometry which varies at each step but remains the same during a single step. Indeed, similarly for the arguments in [9], we construct a natural assumption on the disturbance $\mathbf{w}$ that it does not affect the inner-loop system.

To put it another way, considering the disturbance due to terrain together with the control parameters varying at each step leads to an extended Poincarè map described by

$$\mathbf{x}_{k+1} = \mathbf{P}(\mathbf{x}_k, \mathbf{u}_k, \mathbf{w}_k), \qquad (3)$$

where $\mathbf{u}_k$ and $\mathbf{w}_k$ are the parameters of the controller (2) and the disturbance at k-th step, respectively.

Finally, suppose that the continuous-time state-feedback controller of (2) with a particular set of parameter $\mathbf{u} = \mathbf{u}^*$ constructs a stable limit cycle in the absence of disturbance $\mathbf{w}$, i.e., on the level ground. We call this controller and the

corresponding system the inner-loop controller and the inner-loop system, respectively, throughout the paper. In this sense, the fixed point of the map, which is denoted by $\mathbf{x}^*$, corresponds to the limit cycle in state-space, i.e.,

$$\mathbf{x}^* = \mathbf{P}(\mathbf{x}^*, \mathbf{u}^*, \mathbf{w}^*), \qquad (4)$$

and the (local) orbital stability of the limit cycle is equivalent to the stability of the corresponding fixed point on the Poincarè map [11], [12], [23]. The Poincarè map of the limit cycle walker is often analytically intractable and only found numerically. Therefore, the local orbital stability has been also assessed numerically in [11], [24], and linearizing the Poincarè map and adequately defining the output vector plays a quite important role in such a numerical stability analysis. In connection with this, this paper also develops an optimal control scheme to bounded persistent disturbance on this linearized Poincarè map.

B. REDUCTION TO OPTIMAL CONTROL PROBLEM

Because the map $\mathbf{P}$ can be assumed to be differentiable with respect to $\mathbf{x}$, $\mathbf{u}$, $\mathbf{w}$ at the fixed point (see [12] for details), we can conduct the linearization of the nonlinear discrete-time dynamical system of (3), by which we can lead to the discrete-time linear time-invariant system described by

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}_1\mathbf{w}_k + \mathbf{B}_2\mathbf{u}_k, \qquad (5)$$

where $\mathbf{A} = \frac{\partial \mathbf{P}}{\partial \mathbf{x}}|_{(\mathbf{x}^*,\mathbf{u}^*,\mathbf{0})}$, $\mathbf{B}_1 = \frac{\partial \mathbf{P}}{\partial \mathbf{w}}|_{(\mathbf{x}^*,\mathbf{u}^*,\mathbf{0})}$, and $\mathbf{B}_2 = \frac{\partial \mathbf{P}}{\partial \mathbf{u}}|_{(\mathbf{x}^*,\mathbf{u}^*,\mathbf{0})}$. Without loss of generality, we further assume that the state vectors, inner-loop control parameters and as well as disturbances are chosen such that $(\mathbf{x}^*, \mathbf{u}^*, \mathbf{w}^*) = (\mathbf{0}, \mathbf{0}, \mathbf{0})$.

As mentioned before, the main objective of this paper is to improve the performance of the inner-loop system (5) for the disturbance $\mathbf{w}$, but there is no method for achieving such an objective if we are confined ourselves to this system description (i.e., (5)). Thus, it is quite important to construct the corresponding problem formulation, and this situation motivates us to develop an involved approach to a hierarchical control architecture consisting of the inner-loop system (i.e., (5)) together with an outer-loop system, which adjusts the inner-loop control parameter $\mathbf{u}$. As a solution for this problem, in particular, this paper proposes to adopt the optimal linear control design formulation.

We begin with noting that the n_z-dimensional regulated output $\mathbf{z} = \mathbf{h_1}(\mathbf{x}, \mathbf{u}, \mathbf{w})$ and the n_u-dimensional measurement output $\mathbf{y} = \mathbf{h_2}(\mathbf{x}, \mathbf{u}, \mathbf{w})$ could be properly defined according to the desired performance objective and the practical conditions of the implemented sensors of the bipedal robot. In this regard, we consequently define the sensitivity matrices as $\mathbf{C}_1 = \frac{\partial \mathbf{h_1}}{\partial \mathbf{x}}|_{(\mathbf{x}^*,\mathbf{u}^*,\mathbf{0})}$, $\mathbf{D}_{11} = \frac{\partial \mathbf{h_1}}{\partial \mathbf{w}}|_{(\mathbf{x}^*,\mathbf{u}^*,\mathbf{0})}$, $\mathbf{D}_{12} = \frac{\partial \mathbf{h_1}}{\partial \mathbf{u}}|_{(\mathbf{x}^*,\mathbf{u}^*,\mathbf{0})}$. Here, similarly for the case of $\mathbf{P}$, $\mathbf{h_1}$ and $\mathbf{h_2}$ are assumed to be differentiable with respect to $\mathbf{x}$, $\mathbf{u}$, $\mathbf{w}$ at the fixed point. The output associated with the fixed point is also

described by $\mathbf{z}^* = \mathbf{h}_1(\mathbf{x}^*, \mathbf{u}^*, \mathbf{0}) = \mathbf{0}$. Indeed, the measurement $\mathbf{y}$ would be adequately determined to be used for the feedback controller design; the sensitivity matrices $\mathbf{C}_2$, $\mathbf{D}_{21}$ and $\mathbf{D}_{22}$ are also defined in an equivalent fashion to those for $\mathbf{z}$.

By combining the above arguments, we can lead to the linearized discrete-time system for $\mathbf{P}$ described by

$$\mathcal{P} : \begin{cases} \mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}_1\mathbf{w}_k + \mathbf{B}_2\mathbf{u}_k \\ \mathbf{z}_k = \mathbf{C}_1\mathbf{x}_k + \mathbf{D}_{11}\mathbf{w}_k + \mathbf{D}_{12}\mathbf{u}_k \\ \mathbf{y}_k = \mathbf{C}_2\mathbf{x}_k + \mathbf{D}_{21}\mathbf{w}_k + \mathbf{D}_{22}\mathbf{u}_k. \end{cases} \quad (6)$$

With this representation in mind, we aim at constructing the problem definition with the consideration of stable walking for uneven terrains. In other words, the problem for analyzing the effect of the disturbance $\mathbf{w}$ on the regulated output $\mathbf{z}$ together with the problem for minimizing this effect should be mathematically formulated. As a preliminary step to proceed the mathematical formulations, we next consider the synthesis of an event-based discrete-time linear dynamic output-feedback controller described by

$$\Psi : \begin{cases} \boldsymbol{\psi}_{k+1} = \mathbf{A}_\psi \boldsymbol{\psi}_k + \mathbf{B}_\psi \mathbf{y}_k \\ \mathbf{u}_k = \mathbf{C}_\psi \boldsymbol{\psi}_k + \mathbf{D}_\psi \mathbf{y}_k, \end{cases} \quad (7)$$

where $\boldsymbol{\psi}_k$ is the state vector of the controller Ψ, $\mathbf{A}_\psi, \mathbf{B}_\psi, \mathbf{C}_\psi, \mathbf{D}_\psi$ are the parameters of the controller to be properly designed. Here, it would be worthwhile to note that it can be expected to achieve an optimal performance for the unknown disturbance $\mathbf{w}$ by employing the extended dynamic output-feedback form as (7) rather than a simple state-feedback form. Furthermore, without loss of generality, the term $\mathbf{D}_{22}$ is assumed to be zero when we consider the synthesis of Ψ and the closed-loop system consisting of $\mathcal{P}$ and Ψ. In this sense, we will assume $\mathbf{D}_{22} = 0$ in the following.

Based on the above two representations of (6) and (7), the stability issues of biped walking on uneven terrains can be summarized as follows (which are also interpreted as generalized versions of Sub-Problem 1 and Sub-Problem 2 in I, respectively):

(I) **Performance analysis problem** With regard to the evaluation of stability for biped walking on uneven terrains, it is quite meaningful to define an adequate quantitative performance measure for the closed-loop system consisting of $\mathcal{P}$ with Ψ, and such a performance measure is expected to be formulated through an induced norm from $\mathbf{w}$ to $\mathbf{z}$. Consequently, a tractable method for computing the performance measure should be developed.

(II) **Optimal controller synthesis problem** In connection with the quantitative performance measure defined above, it is very important to clarify whether or not the synthesis problem of an optimal controller for minimizing the performance measure is solvable. To put it another way, providing a method for designing an optimal controller Ψ which minimizes the performance measure defined for the closed-loop system consisting

of $\mathcal{P}$ with Ψ plays a key role in establishing fundamentals for robust outdoor walking.

III. SOLUTION PROCEDURES

This section is devoted to giving the solution procedures for the problems (I) and (II) discussed at the end of the previous section.

To deal with bounded persistent disturbances together with their appropriate quantitative measure, we first take the l_∞ norm as a topology for signals in the biped walking system. It turns out that the disturbance $\mathbf{w}$ and the regulated output $\mathbf{z}$ are assumed to be in the l_∞ space, and their l_∞ norms are bounded, i.e., $\|\mathbf{w}\|_\infty < \infty$ and $\|\mathbf{z}\|_\infty < \infty$. In this sense, the performance analysis problem illustrated in the previous section can be equivalently restated as the l_1 analysis problem (i.e., the l_∞-induced norm computation problem). At the same line, the optimal controller synthesis problem explained in the previous section can be equivalently restated as the l_1 synthesis problem (i.e., the synthesis problem of the optimal controller minimizing the l_∞-induced norm). It would be worthwhile to note that the analysis/synthesis problem associated with the l_∞-induced norm for a given linear time-invariant (LTI) system is called the l_1 analysis/synthesis problem, because the l_∞-induced norm of a system coincides with the l_1 norm of the impulse response for the single-input/single-output (SISO) case. We further briefly provide the effectiveness of the l_∞-induced norm in biped walking systems as follows.

- If we note that the l_∞-induced norm corresponds to the maximum magnitude of the output derived from the possible worst disturbance, this measure is obviously more practical than other measures in assessing a walking performance of a bipedal robot.
- Disturbances such as changes of slope and height occurred in real biped walking systems can be treated by adopting the l_∞-induced norm. Such disturbances are also physically intuitive since they do not require any specific stochastic characteristics but the maximum amplitude.

The problems corresponding to the l_1 analysis and synthesis which will be tackled in this section can be depicted as Fig. 1.

A. PERFORMANCE ANALYSIS

Let us consider the closed-loop systems consisting of the discrete-time system $\mathcal{P}$ given by (6) and the discrete-time dynamic output feedback controller Ψ given by (7), and denote it by $\mathcal{P}_{cl}$. If we newly define the state of $\mathcal{P}_{cl}$ as $\boldsymbol{\xi}_k := [\mathbf{x}_k^T \ \boldsymbol{\psi}_k^T]^T$, the dynamics of $\mathcal{P}_{cl}$ is described by

$$\mathcal{P}_{cl} : \begin{cases} \boldsymbol{\xi}_{k+1} = \mathbf{A}_{cl}\boldsymbol{\xi}_k + \mathbf{B}_{cl}\mathbf{w}_k \\ \mathbf{z}_k = \mathbf{C}_{cl}\boldsymbol{\xi}_k + \mathbf{D}_{cl}\mathbf{w}_k, \end{cases} \quad (8)$$

where

$$\mathbf{A}_{cl} = \begin{bmatrix} \mathbf{A} + \mathbf{B}_2\mathbf{D}_\psi\mathbf{C}_2 & \mathbf{B}_2\mathbf{C}_\psi \\ \mathbf{B}_\psi\mathbf{C}_2 & \mathbf{A}_\psi \end{bmatrix},$$
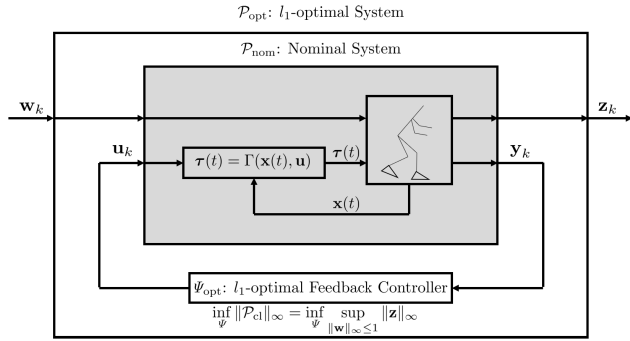
FIGURE 1. A hierarchical control model of biped walking systems. The parameters **u** of the inner-loop feedback controller Γ are updated by an outer-loop event-based feedback controller Ψ. The analysis and synthesis problems of the event-based controller are formulated as the l_1 analysis and synthesis problems on the linearized Poincarè map.

$$\mathbf{B}_{cl} = \begin{bmatrix} \mathbf{B}_1 + \mathbf{B}_2 \mathbf{D}_\psi \mathbf{D}_{21} \\ \mathbf{B}_\psi \mathbf{D}_{21} \end{bmatrix},$$

$$\mathbf{C}_{cl} = \begin{bmatrix} \mathbf{C}_1 + \mathbf{D}_{12} \mathbf{D}_\psi \mathbf{C}_2 & \mathbf{D}_{12} \mathbf{C}_\psi \end{bmatrix},$$

$$\mathbf{D}_{cl} = \begin{bmatrix} \mathbf{D}_{11} + \mathbf{D}_{12} \mathbf{D}_\psi \mathbf{D}_{21} \end{bmatrix}.$$

Here, it is naturally assumed that the discrete-time controller Ψ internally stabilizes the discrete-time system $\mathcal{P}$, i.e., all the eigenvalues of $\mathbf{A}_{cl}$ are in the open unit disk necessarily for the corresponding l_∞-induced norm to be well-defined.

The performance analysis problem tackled in this section is to compute the l_∞-induced norm of the closed-loop system $\mathcal{P}_{cl}$ given by (8), where this induced norm is defined as

$$\|\mathcal{P}_{cl}\|_\infty := \sup_{\|\mathbf{w}\|_\infty \neq 0} \frac{\|\mathbf{z}\|_\infty}{\|\mathbf{w}\|_\infty} = \sup_{\|\mathbf{w}\|_\infty = 1} \|\mathbf{z}\|_\infty. \quad (9)$$

We explain from the above equation the novelty of the l_∞-induced norm as a performance measure for biped walking systems in both theoretical and practical senses as follows. Suppose that a bipedal robot is walking on the rough terrain. Let us further assume that the terrain uncertainties that would affect the dynamics of the robot can be regarded as bounded and persistent disturbance input by noting the fact that the height or slope would vary at every single step the robot interacts with the ground but the change may be bounded by some maximum magnitude. Then, the relation between the bounded persistent disturbance **w** and the regulated output **z** can be quantitatively described by the l_∞-induced norm defined as (9).

It would be worthwhile to note that it is difficult to compute an explicit value of the the the l_∞-induced norm, and thus we provide a simple and approximate (but asymptotically exact) method for obtaining the induced norm with any degree of accuracy, which can be also interpreted as one of the contributions in this paper. To this end, we introduce the truncation parameter $N \in \mathbb{N}_0$ and the matrices P_{cl} and P_{Ncl}^- defined respectively as

$$P_{cl} := \begin{bmatrix} \mathbf{D}_{cl} & \mathbf{C}_{cl}\mathbf{B}_{cl} & \mathbf{C}_{cl}\mathbf{A}_{cl}\mathbf{B}_{cl} & \cdots \end{bmatrix} \quad (10)$$

$$P_{Ncl}^- := \begin{bmatrix} \mathbf{D}_{cl} & \mathbf{C}_{cl}\mathbf{B}_{cl} & \cdots & \mathbf{C}_{cl}\mathbf{A}_{cl}^N\mathbf{B}_{cl} \end{bmatrix} \quad (11)$$

Here, the finite-dimensional matrix P_{Ncl}^- is interpreted as a truncated version of the infinite-dimensional matrix P_{Ncl}. With these matrices, we introduce the following results.

Lemma 1: The following equation holds:

$$\|\mathcal{P}_{cl}\|_\infty = \|P_{cl}\|_\infty. \quad (12)$$

Lemma 2: There exists a constant $M \in \mathbb{N}_0$ such that

$$\|\mathbf{A}_{cl}^M\|_\infty < 1 \quad (13)$$

With such an M, define the vector $\alpha_N^{[M]}$ as

$$\alpha_N^{[M]} := \begin{bmatrix} \alpha_{N,1}^{[M]} \\ \vdots \\ \alpha_{N,n_z}^{[M]} \end{bmatrix} \quad (14)$$

where

$$\alpha_{N,i}^{[M]} := \frac{\|\mathbf{C}_{cl,i}\mathbf{A}_{Ncl}^{[M]}\|_\infty \cdot \|\mathbf{B}_{cl}\|_\infty}{1 - \|\mathbf{A}_{cl}^M\|_\infty}, \quad (15)$$

$$\mathbf{A}_{Ncl}^{[M]} := \begin{bmatrix} \mathbf{A}_{cl}^{N+1} & \cdots & \mathbf{A}_{cl}^{N+M} \end{bmatrix} \quad (16)$$

and $\mathbf{C}_{cl,i}$ is the ith row vector of $\mathbf{C}_{cl}$. Then, we obtain the inequality

$$\|P_{cl}\|_\infty \leq \|P_{Ncl}^{[MU]}\|_\infty, \quad (17)$$

where

$$P_{Ncl}^{[MU]} := \begin{bmatrix} P_{Ncl}^- & \alpha_N^{[M]} \end{bmatrix}. \quad (18)$$

Lemma 3: Define the vector β_N as

$$\beta_N := \begin{bmatrix} \beta_{N,1} \\ \vdots \\ \beta_{N,n_z} \end{bmatrix}, \quad (19)$$

where

$$\beta_{N,i} := (\mathbf{C}_{cl,i}\mathbf{A}_{cl}^{N+1}X_{cl}(\mathbf{A}_{cl}^T)^{N+1}\mathbf{C}_{cl,i}^T)^{1/2} \quad (20)$$

and X_{cl} is obtained through the following discrete-time Lyapunov equation:

$$\mathbf{A}_{cl}X_{cl}\mathbf{A}_{cl}^T - X_{cl} + \mathbf{B}_{cl}\mathbf{B}_{cl}^T = 0. \quad (21)$$

Then, we obtain the inequality

$$\|P_{Ncl}^{[L]}\|_\infty \leq \|P_{cl}\|_\infty, \quad (22)$$

where

$$P_{Ncl}^{[L]} := \begin{bmatrix} P_{Ncl}^- & \beta_N \end{bmatrix}. \quad (23)$$

The proofs of Lemmas 1–3 are provided in the appendix since they are quite technical. Lemma 1 is relevant to the analytic computation of $\|\mathcal{P}_{cl}\|_\infty$ but it involves an infinite-dimensional matrix and thus such an analytic computation is a non-trivial task. In this respect, the approximate methods for computing an upper bound and a lower bound of $\|\mathcal{P}_{cl}\|_\infty$ are introduced in Lemma 2 and Lemma 3, respectively. We readily obtain from Lemmas 1–3 the following result.

Theorem 1: The following inequality holds:

$$\|P_{Ncl}^{[L]}\|_\infty \le \|\mathcal{P}_{cl}\|_\infty \le \|P_{Ncl}^{[MU]}\|_\infty. \qquad (24)$$

Indeed, the gap between the upper and lower bounds in (24) converges to 0 at an exponential order of N.

Proof: The first assertion is obvious from Lemmas 1–3. The second assertion is also readily followed by noting that $\|\mathbf{A}_{Ncl}^{[M]}\|_\infty$ and $\|\mathbf{A}_{cl}^N\|_\infty$ (as well as $\alpha_N^{[M]}$ and β_N) tend to 0 at an exponential order of N because $\mathbf{A}_{cl}$ is stable. ∎

Theorem 1 implies that we can compute $\|\mathcal{P}_{cl}\|_\infty$ within an arbitrary degree of accuracy as the corresponding parameter N becomes larger. Hence, we can conduct the quantitative stability analysis for biped walking on uneven terrains by using the arguments of Theorem 1.

B. OPTIMAL CONTROLLER SYNTHESIS

Even though the arguments of Theorem 1 are quite useful to analyze the effectiveness of a given event-based controller as well as the robustness of the overall closed-loop system (i.e., the biped walking system) for unknown bounded persistent disturbances (such as uneven terrains), this theorem is limited to the performance analysis problem and cannot be employed in the corresponding controller synthesis problem by itself. In this sense, we introduce a method for designing an optimal event-based controller for minimizing $\|\mathcal{P}_{cl}\|_\infty$, i.e., the l_1 synthesis for $\|\mathcal{P}_{cl}\|_\infty$.

The l_1 synthesis can be explained as finding an optimal controller Ψ which minimizes the l_∞-induced norm $\|\mathcal{P}_{cl}\|_\infty$, and it would be mathematically represented by

$$\inf_\Psi \|\mathcal{P}_{cl}\|_\infty = \inf_\Psi \sup_{\|\mathbf{w}\|_\infty \le 1} \|\mathbf{z}\|_\infty =: \|\mathcal{P}_{opt}\|_\infty \qquad (25)$$

Denote an optimal controller Ψ achieving the infimum in (25) by Ψ_{opt}. Then, the l_1 synthesis problem is to find Ψ_{opt}, and this can be interpreted as a sort of min-max problems as shown in (25).

Even though the details are omitted for a limited space, we briefly introduce the solution procedure for the l_1 synthesis problem given by (25). The l_1 synthesis problem could be equivalently represented by a linear programming (LP) problem, but it intrinsically involves a non-trivial property of infinite dimensions. To alleviate the difficulties originated from such a property, the duality theorem [25] is used in [18], [19] to transform the primal problem into its dual problem. In other words, these studies develop approximate methods such as truncation idea and delay augmentation, by which one could achieve an optimal controller with an arbitrary degree of accuracy by solving the corresponding approximate versions iteratively. In this sense, the problem of optimal controller synthesis discussed in this paper is solvable through an LP problem, and thus we could expect an optimal performance of the biped walking system for uneven terrains by using the optimal controller.

More precisely, regarding the l_1 optimal controller as the event-based parameter update law, the chosen parameters $\mathbf{u}$ of the inner-loop controller can be updated at each step in such a manner that the worst $\mathbf{z}$, deviation of the system output from the fixed point on the Poincaré map, is minimized while walking on a terrain with unknown geometry which varies at each step $\mathbf{w}$. This method is undoubtedly expected to improve the actual performance for suppressing disturbances of the system by minimizing the risk of fall.

IV. COMPARISON TO EXISTING MEASURES

This section is concerned with a comparison of the new candidate of the l_∞-induced norm to other measures used in biped walking systems. We first provide several practical advantages over the other measures considered on the Poincaré map, such as the largest Floquet multiplier, H_2 norm, and H_∞ norm. We also introduce other measures irrelevant to the Poincaré map, which are also used in biped walking systems, and consider comparison to the l_∞-induced norm proposed in this paper.

A. MEASURES ASSOCIATED WITH THE POINCARÈ MAP

The following measures listed in this subsection are associated with the Poincaré map for biped walking systems, but there exist some disadvantages for robust walking on uneven terrains.

1) LARGEST FLOQUET MULTIPLIER OF A LIMIT CYCLE

The eigenvalues defined on the corresponding linearized Poincarè map of a limit cycle is called characteristic or Floquet multipliers [23]. Because estimating approximate values of the Floquet multipliers of stable limit cycle walkers in experiments and in numerical simulations is computationally not expensive, they have been widely used in the stability analysis for both robot and human walking as discussed in [11], [24], [26], [27]. In particular, the largest Floquet multiplier contains information of decaying rate of small perturbation over time, thus it has been used for disturbance rejection measure. However, it was revealed in [16] that there exists only small correlation between the actual disturbance rejection performance and the modulus of the maximum eigenvalue. Indeed, the information encoded in the Floquet multipliers is not enough to fully characterize the response of the system to the external disturbances.

2) THE H_2 NORM

The gait sensitivity norm (GSN) is introduced in [16] to provide a measure of the dynamic response of gait indicators to a disturbance, and it is essentially equivalent to the H_2 norm defined on the linearized Poincaré map of the gait system. To put it another way, the GSN (i.e., the H_2 norm) corresponds to the standard deviation of the system output response to white noise as well as unit impulse disturbances. This measure showed high correlation to actual disturbance rejection performance, and it is computationally efficient to obtain the GSN in both simulations and experiments. Due to its advantages, the GSN is also used in robotics and human gait analysis [28], [29]. In spite of the aforementioned advantages of the GSN, there could exist practical limitations arising

from the fact that the GSN is related to averaged performance for the impulse disturbance of the system, compared to the advantages of the measure proposed in this paper (see III).

3) THE H_∞ NORM

The H_∞ norm of a system corresponds to the maximum energy of the output for the energy-bounded worst input, i.e., the L_2-induced norm of the system. Hence, the fact that it is related with the worst case performance for the energy-bounded disturbances could make it more practical than the GSN when we consider bipedal robots on rough terrains. In this sense, a systematic approach to design an optimal controller for stabilizing the periodic orbit of the system together with minimizing the H_∞ norm in the presence of a single step-up/down disturbance was presented in [7]. Even though it is tempting to use the H_∞ norm as a measure in the presence of such a disturbance (i.e., single step-up/down and it can be also equivalently regarded as an impulse response), it is still questionable to validate the effect of rough terrains on bipedal robots as the single step-up/down disturbance. More importantly, it is intrinsically assumed in the H_∞ norm that disturbances have bounded energy and they are regarded as decreasing functions of time, but such an assumption cannot be generally constructed on practical rough terrains of biped walking systems.

B. OTHER MEASURES IRRELEVANT TO THE POINCARÈ MAP

In this section, we introduce various existing performance measures for quantifying the disturbance rejection of biped walking systems. In particular, we are concerned with the limitations of the following measures in terms of how to utilize them to improve the walking performance on rough terrains.

1) GENERIC MEASURES OF STABILITY TO 'AVOID FALLING'

If one can find a set of points in the state-space from which a bipedal robot is guaranteed to be able to avoid falling, it should be considered as the most generic measure of stability. In connection with this, the 'viability kernel' proposed in [30] captures this concept mathematically. Similarly, the basin of attraction (BoA) is the set of all states that converges to the limit cycle asymptotically. These sets require an evaluation of the full nonlinear system behavior at all admissible state-space to predict its behavior as time grows. Given any disturbance, whether the states of a bipedal robot are in these sets or not can be used to determine whether the bipedal robot can reject the disturbance. Consequently, these measures could be interpreted as exact or even ultimate criteria to analyze the system performance against disturbances. However, obtaining these notions of stability is computationally very expensive in numerical simulations and even more cumbersome in real experiments since they require a full forward nonlinear dynamic simulation or actual experiments from all possible states. Therefore, they have a limited practical value for control synthesis.

2) LARGEST ALLOWABLE DISTURBANCE

The maximum size of the disturbances that the system can manage has been used as a measure for disturbance rejection performance. This could be quantified by gradually increasing the magnitude of a specific type of disturbance until a robot eventually fails to walk. The type of disturbances may include a pushing force at the hip [31], slope of the ground [32], and a step-down height [33]. Even though the measures give useful information, it is problematic to adjust or re-design a controller based on it since they provide no insight regarding the effect of the control parameter on the maximum magnitude that a bipedal robot can manage. Moreover, they do not provide any information on sequential disturbances which are typically expected when walking on rough grounds.

3) CRITERIA FOR SUSTAINED LOCAL STABILITY

For ease of a controller synthesis, local static and dynamic stability criteria have been developed in [34], [35], where the center of mass (CoM; for static) or the zero moment point (ZMP; for dynamic) is required to be kept within base of support of a bipedal robot at all time. Computation of the margin of stability for these criteria is formal and efficient. At a similar line, computing the maximal output admissible (MOA) set within finite time has been conducted [36], [37], in which time series of a control input are generated to satisfy the ZMP stability constraint. However, this process requires a constraint such that at least one foot of a bipedal robot should be firmly placed on the ground, and thus its application to a broader range of biped walking systems such as underactuated systems is a non-trivial task.

V. SIMULATION RESULTS

In this section, we exploit one of representative models of biped walking to validate the proposed method with the event-based l_1 optimal control. We also use the notation $\mathcal{P}_{nom}$ to denote a nominal system consisting of a Poincarè map (given by (1)) and a state-feedback controller with fixed parameters (given by (2)), i.e.,

$$\mathcal{P}_{nom} : \begin{cases} \mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}_1\mathbf{w}_k \\ \mathbf{z}_k = \mathbf{C}_1\mathbf{x}_k + \mathbf{D}_{11}\mathbf{w}_k \end{cases}. \tag{26}$$

This is essentially equivalent to substituting $\mathbf{u}_k = \mathbf{0}$ in (6), or more precisely, $\mathbf{A}_\psi = \mathbf{0}$, $\mathbf{B}_\psi = \mathbf{0}$, $\mathbf{C}_\psi = \mathbf{0}$ and $\mathbf{D}_\psi = \mathbf{0}$ in (7), i.e., no change in control parameters.

A. MODEL DESCRIPTION

We adopt the state-determined biped walking model [27], in which the energy balance between dissipation (foot-ground collision) and compensation (ankle actuation) makes the corresponding limit cycle to be asymptotically stable. In this planar model, complex mechanics of walking is deliberately simplified so that the model has only one degree of freedom.[1]

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[1]For more details, readers are referred to the original article [27].
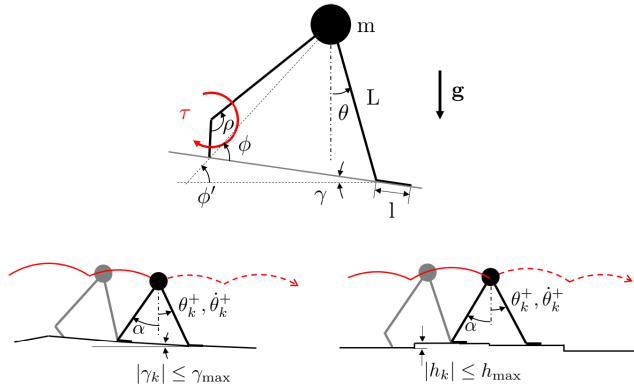
FIGURE 2. The simple walking model on unstructured terrains on which terrain slope (bottom left) or height (bottom right) varies at each step.

Because the Poincarè map is analytically tractable, this simple model is quite useful to validate the proposed method with the event-based l_1 optimal control. We deal with the case when this model walks on the unstructured terrains of which slope γ varies at each step while its variation is bounded by $|\Delta\gamma| = |\gamma| \leq \Delta\gamma_{\max}$, as illustrated in Fig. 2. Model parameters including point mass m and lengths of rigid and massless legs and feet L and l, are listed in Table 1.

The model has two phase: single stance and double stance. In single stance phase, the model behaves as a passive inverted pendulum pivoting around the leading heel. During the double stance phase, the ankle of the trailing leg behaves like a torsional spring and generates torque τ as

$$\tau = \kappa(\mu - \rho), \quad \frac{\pi}{2} - \alpha \leq \rho \leq \mu, \qquad (27)$$

where μ is the maximal plantar ankle extension, ρ is the ankle angle, and κ is the proportional gain. Hence, the equation of motion becomes

$$mL^2\ddot{\theta} = mgL\sin\theta - \frac{\sqrt{L^2 + l^2 - 2Ll\cos\rho}}{l\sin\rho}\cos(\theta - \phi')\tau,$$

where intermediate variables ϕ' and ρ are obtained from geometry. When ankle torque becomes zero, it is followed by take-off of the trailing leg and smooth transition from the double to the single stance phase. Transition from the single to the double stance phase occurs with the inelastic and instantaneous collision, which introduces energy loss and discrete jump in states as

$$\dot{\theta}^+ = \cos(2\alpha)\dot{\theta}^-,$$

where $\dot{\theta}^+$ and $\dot{\theta}^-$ are the post- and pre-collision angular velocities. Here, it is assumed that the inter-leg angle at touchdown is controlled to be 2α.

B. PROBLEM FORMULATION

Taking the moment right after the foot-ground collision as the Poincarè section, one can obtain an analytic Poincarè map from the energy conservation and the inelastic impulsive

TABLE 1. Parameters of the simple walking model [27].

Parameter	Meaning	Value
m	mass	80 kg
L	leg length	1 m
l	foot length	0.2 m
g	gravitational acceleration	9.81 m/s^2
α	angle of the leg at heel strike	$\pi/6$ rad
μ	maximal plantar extension of the ankle	2.6 rad
γ^*	slope (nominal value)	0 rad
κ^*	ankle actuation gain (nominal value)	87.3 N·m/rad

energy loss as follows

$$\dot{\theta}_{k+1}^+ = -\cos(2\alpha)\sqrt{(\dot{\theta}_k^+)^2 + 2\frac{\Delta E_{grav} + E_{in}}{mL^2}}, \qquad (28)$$

where the subscript k represents k-th step, $\Delta E_{grav} = mgL(\cos(\alpha - \gamma_{k-1}) - \cos(\alpha + \gamma_k))$ is the gravitational potential energy change, and $E_{in} = \int \tau d\theta = \frac{1}{2}\kappa(\mu - (\frac{\pi}{2} - \alpha))^2$ is the energy injected per step by ankle actuation.

The state and disturbance of the system are defined as $\mathbf{x}_k := \dot{\theta}_k^+$, $\mathbf{w}_k := \begin{bmatrix} \Delta\gamma_{k-1}, \Delta\gamma_k \end{bmatrix}^T$, and the output and measurement of the system are selected to be the angular velocity at the beginning of the double stance phase as $\mathbf{z}_k = \mathbf{y}_k := \dot{\theta}_k^+$. This is a natural choice to evaluate performance of biped walking, because insufficient kinetic energy will result in failure of the model to vault over and make the next step. The inner-loop controller (27) is modified as

$$\tau = (\kappa_* + \Delta\kappa)(\mu - \rho), \quad \frac{\pi}{2} - \alpha \leq \rho \leq \mu, \qquad (29)$$

such that the parameter, torsional spring constant of the ankle, is adjusted at each step by the optimal controller (25) with the choice of $\mathbf{u}_k := \Delta\kappa_k$.

C. NUMERICAL EVALUATION

We evaluate the performance of the l_1 optimal system and compared it with that of the nominal system on the same uneven terrain. The l_∞-induced norms of the nominal system $\mathcal{P}_{nom}$ and the optimal system $\mathcal{P}_{opt}$ are also compared with the actual value $\|\mathbf{z}\|_\infty$, the l_∞-norm of the sequence of outputs, measured from numerical simulations. To do this, we first generate random terrains on which the models attempted to walk for a hundred steps. The slope of the terrain varies for each step, but the variation is bounded by a given number $\|\mathbf{w}\|_\infty = \Delta\gamma_{\max}$. The simulations are conducted for a range of $\|\mathbf{w}\|_\infty$, and for each $\|\mathbf{w}\|_\infty$, multiple random terrains are generated. For both the nominal and optimal systems, it is considered failure when the point mass failed to vault over and make the next step, and the success rate is computed by counting the frequency of successful ones of all trials. In order to compare the proposed optimal controller with other methods such as LQR [13], the H_2 [15] and the H_∞ [16], success rate of systems equipped with each controller is measured. Numerical simulation is conducted with MATLAB (Mathworks, MA, USA) with integrator `ode45`.

The behavior of the nominal system and the l_1 optimal system in response to uneven terrains with the maximum slope
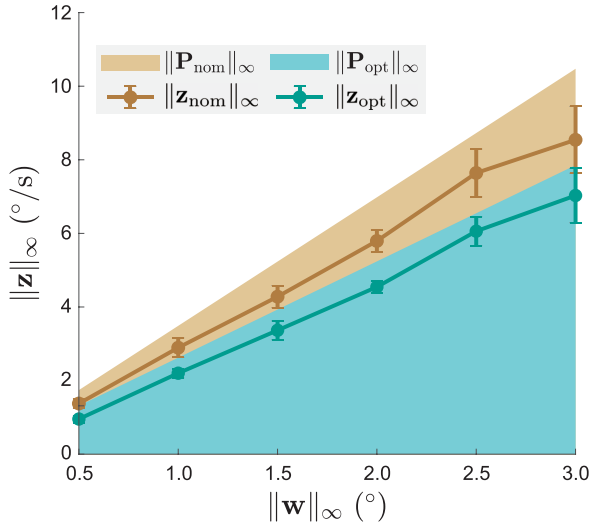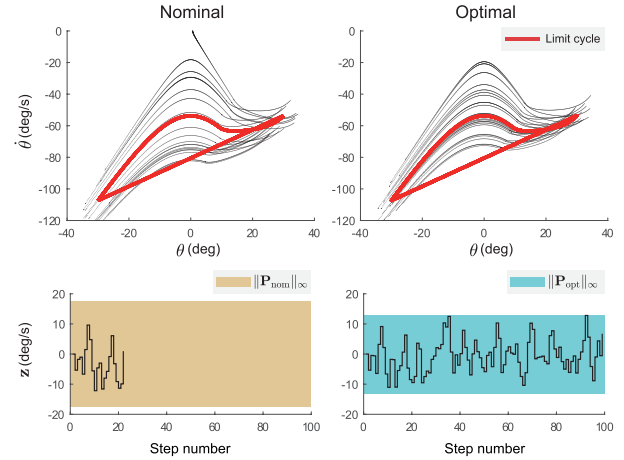
FIGURE 3. The worst system output computed from l_∞-induced norm of the system for nominal and optimal system ($\|\mathcal{P}_{nom}\|_\infty$ and $\|\mathcal{P}_{opt}\|_\infty$, respectively) and the worst output measured from numerical dynamic simulation ($\|\mathbf{z}_{nom}\|_\infty$ and $\|\mathbf{z}_{opt}\|_\infty$, respectively) over $\|\mathbf{w}\|_\infty \in [0, 3]$ deg. $\|\mathbf{z}\|_\infty$ are represented as mean and standard deviations of ten trials for each condition.

FIGURE 4. Simulation results of the nominal and the optimal system over $\|\mathbf{w}\|_\infty = 5$ deg. Top: system phase portrait (black, dotted line: trajectory of models on rough terrain with $\|\mathbf{w}\|_\infty = 5°$, red, solid line: limit cycle on level terrain). Bottom: system output at each step. Shaded regions illustrate the l_∞-induced norm $\|\mathcal{P}_{nom}\|_\infty$ and $\|\mathcal{P}_{opt}\|_\infty$ for the nominal and the optimal system, respectively.

change $\|\mathbf{w}\|_\infty \in [0, 3]$ deg are presented in Fig. 3. While both the nominal and the optimal system can successfully walk hundred steps in this range of disturbances, the proposed method successfully reduces the system variations during walking as expected. Moreover, actual worst system outputs $\|\mathbf{z}\|_\infty$ are well-bounded by the computed l_∞-induced norms in both the nominal and the optimal system (i.e., $\|\mathcal{P}_{nom}\|_\infty$ and $\|\mathcal{P}_{opt}\|_\infty$). These results indicate that the proposed event-based l_1-optimal control can perform as effective tool for robust walking on rough terrain.

The effect of the proposed method is more clear in Fig. 4, where the maximum slope change is $\|\mathbf{w}\|_\infty = 5$ deg. The nominal system fails after 23 steps; the failure is shown as the system trajectory approaches to $\dot{\theta} = 0$ when $\theta > 0$ in the phase portrait. This physically implies that the energy injected by the ankle actuation is not enough to compensate for the dissipated energy. Hence, the system cannot vault over to make next step. In contrast, the event-based l_1 optimal controller updates κ at each step such that the ankle propulsion adapts energy injection as needed so that the model completes hundred steps successfully. Furthermore, the results verifies that $\|\mathbf{z}\|_\infty$ are well-bounded by the computed l_∞-induced norms $\|\mathcal{P}\|_\infty$. These observations obviously demonstrate that system disturbance rejection is significantly improved by the event-based l_1-optimal control.

In order for comparison with other methods associated with the Poincarè map, we conduct extensive simulations for various $\|\mathbf{w}\|_\infty \in [3, 6]$ deg. In these simulations, we randomly generate twenty different trials and compute the success rate by the number of trials that the model completes hundred steps out of the trials. The success rates of the nominal and l_1 optimal systems together with those with different types of event-based feedback controllers for each $\|\mathbf{w}\|_\infty$ are presented in Table 2. It is obvious from Table 2 that the

proposed method with the event-based l_1 optimal controller leads to higher success rates than the nominal system and the methods with the H_2 and LQR. More importantly, the method proposed in this paper makes the system successfully walks on rough terrains at $\gamma_{max} = 4.8$[deg], while the nominal system and the methods with the event-based H_2 optimal controller and the event-based LQR show quite low success rates. Hence, this tendency clearly implies that the method with the l_1 optimal event-based controller can be obviously regarded as outperforming the methods with the event-based H_2 optimal controller and the event-based LQR (as well as the nominal system) in biped walking on rough terrains. On the other hand, in the comparison between the methods with the event-based l_1 optimal and H_∞ optimal controllers, we can observe that they have the completely same success rates for all $\|\mathbf{w}\|_\infty$[deg] and thus one might not conclude from the simulation results that the method with the l_1 optimal event-based controller is superior to that with the H_∞ optimal event-based controller. This observation can be interpreted as arising from the fact that both the resulting event-based controllers (i.e., the l_1 and H_∞ optimal controllers) have numerically similar structures since the nominal system used in the simulations is represented by a sort of single-input/single-output (SISO) positive systems. Hence, the tendency that the methods with the l_1 and H_∞ optimal event-based controllers derive the same numerical results associated with the success rates is a natural consequence of our numerical example. More sophisticated comparison between these two methods through numerical simulations is left for an interesting future topic.

VI. APPLICATION: FAILURE PREDICTION

This section considers an extensive issue on possibility of the proposed method to failure prediction. In this regard, we use

TABLE 2. Success rate (%) of different systems over $\|\mathbf{w}\|_\infty$.

	$\|\mathbf{w}\|_\infty$ [deg]					
	3	3.6	4.2	4.8	5.4	6
Nominal	100	45	0	0	0	0
H_2 optimal	100	65	5	0	0	0
LQR	100	100	100	10	0	0
H_∞ optimal	100	100	100	100	5	0
l_1 optimal	100	100	100	100	5	0

the simple walking model given in the preceding section but now terrain height at each step varies ($h_k < h_{\max}$), not slope; it should be remarked that the proposed framework can be generalized to any type of (sequential) terrain disturbance as long as it stays bounded.

From energy balance equation, one can derive threshold post-collision velocity ($\dot{\theta}_{th}^+$) of the model analytically,

$$\frac{1}{2}m(L\dot{\theta}_{th}^+)^2 + mgL\cos(\alpha - \gamma) + \tilde{E}'_{in} = mgL, \quad (30)$$

where $\tilde{E}'_{in} = \int_{\alpha-\gamma}^0 \tau(\theta)\,d\theta$ and $2L\sin\alpha\sin\gamma = -h$ (see [20] for details). If the kinetic energy of the model after collision is smaller than the corresponding threshold value, the model fails to vault over and move forward.

One can numerically solve (30) to express $\dot{\theta}_{th}^+$ as a function of h. The intersection between the analytic threshold velocity and $\|\mathcal{P}_{nom}\|_\infty$ computed for a given $\|\mathbf{w}\|_\infty = h_{\max}$ will determine the threshold height h_{th}. From this result, one can predict that the model can overcome unknown disturbance bounded by $h_{\max} < h_{th}$ but would be likely to fail when $h_{\max} \geq h_{th}$. This result is illustrated in Fig. 5. Graphically the threshold height for the nominal system is determined to be $h_{th,nom} \sim 0.043$ m.

Since the proposed measure represents the worst performance one may expect, $\|\mathcal{P}_{nom}\|_\infty > \dot{\theta}_{th}^+$ should not be considered as a necessary condition to fall; rather, one may consider $\|\mathcal{P}_{nom}\|_\infty < \dot{\theta}_{th}^+$ as a sufficient condition not to fall. Indeed, simulation experiment reveals that the model never falls with $h_{\max} \leq 0.04$ (m), rarely falls with $h_{\max} = 0.045$ (m), and often falls with $h_{\max} \geq 0.05$ (m). Furthermore, the proposed event-based l_1 optimal controller increases the value of $h_{th,opt}$ to ~ 0.048 [m]. With this small change, the difference is evident.

In practice, analytic expression of falling over is rarely obtainable. Instead, one may *design* failure criteria as a function of disturbance. If the failure threshold is properly designed, then the proposed method allows for predicting and maximum terrain roughness on which the bipedal robot would fail as well as quantifying improvement by the optimal control. In other words, the method allows to estimate stable region for the robot from falling through such a procedure. Moreover, if inner-loop system is designed with a stability criteria such as ZMP as in [37], introduction of the l_∞-induced norm and design of optimal controller would be more straightforward.
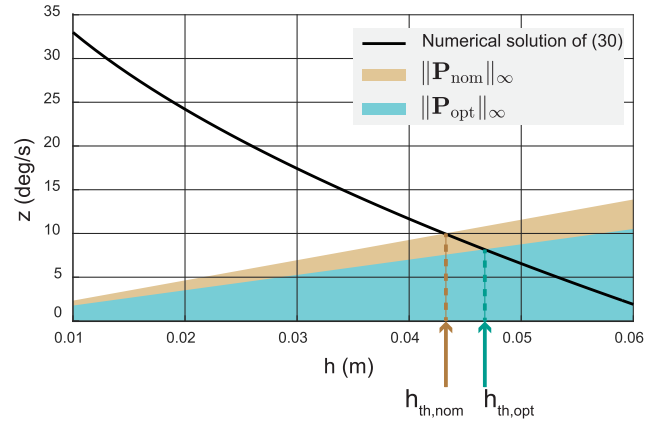
FIGURE 5. The threshold post-collision velocity (black, solid), l_∞-induced norm of the nominal and the optimal system ($\|\mathcal{P}_{nom}\|_\infty$ and $\|\mathcal{P}_{opt}\|_\infty$, respectively). The intersection of the l_∞-induced norm and the threshold post-collision velocity determines h_{th} graphically.

VII. CONCLUSION

The main contribution of this paper is to introduce a systematic treatment of improving performance of bipedal robot walking on rough terrains of which geometry varies every step with some bounded magnitude. More precisely, we suggest the l_∞-induced norm defined on the linearized Poincarè map as a performance analysis measure *for the first time*, and proposed the event-based l_1 optimal controller to adjust parameters of inner-loop controller for minimizing the performance measure. Furthermore, numerical simulations in this paper demonstrated the effectiveness and validity of the proposed performance measure and the optimal controller.

It should be noted that the disturbance treated by this method is limited to those only discretely change at each footstep and of which influence remains the same while the foot is in contact with the ground. Other classes of disturbances such as continuous-time noises and parametric uncertainties should be dealt with by other appropriate means. Applications of the proposed method to higher degrees-of-freedom models and real platforms would be interesting future work. In such application procedures, the approximation errors derived from linearization of the nonlinear Poincarè map should be rigorously treated. Extending the proposed measure to design a high-level continuous-time feedback controller as in [4], [7], [8], [38] would also be one of interesting future directions.

APPENDIX

This appendix provides the proofs of the lemmas given in this paper.

A. PROOF OF LEMMA 1

Note that the input-output relation of $\mathcal{P}_{cl}$ (given by (8)) is described by

$$\mathbf{z}_k = \sum_{i=0}^{k-1} \mathbf{C}_{cl}\mathbf{A}_{cl}^{k-1-i}\mathbf{B}_{cl}\mathbf{w}_i + \mathbf{D}_{cl}\mathbf{w}_k.$$

This together with the definition of the l_∞-induced norm provided in the Introduction leads to

$$\sup_{\|\mathbf{w}\|_\infty \leq 1} \|\mathbf{z}_k\|_\infty$$
$$= \| \begin{bmatrix} \mathbf{D}_{cl} & \mathbf{C}_{cl}\mathbf{B}_{cl} & \mathbf{C}_{cl}\mathbf{A}_{cl}\mathbf{B}_{cl} & \cdots & \mathbf{C}_{cl}\mathbf{A}_{cl}^{k-1}\mathbf{B}_{cl} \end{bmatrix} \|_\infty. \tag{31}$$

Because the right-hand-side (RHS) of (31) is obviously increasing as k becomes larger, it immediately follows that

$$\max_{k \in \mathbb{N}_0} \sup_{\|\mathbf{w}\|_\infty \leq 1} \|\mathbf{z}_k\|_\infty$$
$$= \| \begin{bmatrix} \mathbf{D}_{cl} & \mathbf{C}_{cl}\mathbf{B}_{cl} & \mathbf{C}_{cl}\mathbf{A}_{cl}\mathbf{B}_{cl} & \mathbf{C}_{cl}\mathbf{A}_{cl}\mathbf{B}_{cl} & \cdots \end{bmatrix} \|_\infty.$$

This completes the proof.

B. PROOF OF LEMMA 2

It readily follows from the stability assumption of $\mathbf{A}_{cl}$ that $\|\mathbf{A}_{cl}^k\|_\infty \to 0$ as $k \to \infty$, and thus there exists a constant $M \in \mathbb{N}_0$ of (13). Hence, the first assertion is proved.

For the proof of the second assertion, note that

$$\| \begin{bmatrix} \mathbf{C}_{cl,i}\mathbf{A}_{cl}^{N+1}\mathbf{B}_{cl} & \mathbf{C}_{cl,i}\mathbf{A}_{cl}^{N+2}\mathbf{B}_{cl} & \cdots \end{bmatrix} \|_\infty$$
$$\leq \| \begin{bmatrix} \mathbf{C}_{cl,i}\mathbf{A}_{cl}^{N+1} & \mathbf{C}_{cl,i}\mathbf{A}_{cl}^{N+2} & \cdots \end{bmatrix} \|_\infty \cdot \|\mathbf{B}_{cl}\|_\infty$$
$$\leq \frac{\|\mathbf{C}_{cl,i}\mathbf{A}_{Ncl}^{[M]}\|_\infty \cdot \|\mathbf{B}_{cl}\|_\infty}{1 - \|\mathbf{A}_{cl}^M\|_\infty} = \alpha_{N,i}^{[M]}. \tag{32}$$

This together with the definition of the matrix ∞-norm obviously completes the proof because

$$P_{cl} = \begin{bmatrix} P_{Ncl}^- & \mathbf{C}_{cl}\mathbf{A}_{cl}^{N+1}\mathbf{B}_{cl} & \mathbf{C}_{cl}\mathbf{A}_{cl}^{N+2}\mathbf{B}_{cl} & \cdots \end{bmatrix}. \tag{33}$$

C. PROOF OF LEMMA 3

We first note that $\{\mathbf{w} \mid \|\mathbf{w}\|_2 \leq 1\} \subset \{\mathbf{w} \mid \|\mathbf{w}\|_\infty \leq 1\}$, where $\|\cdot\|_2$ means the standard l_2 norm of a vector sequence. It then readily follows that

$$\| \begin{bmatrix} \mathbf{C}_{cl,i}\mathbf{A}_{cl}^{N+1}\mathbf{B}_{cl} & \mathbf{C}_{cl,i}\mathbf{A}_{cl}^{N+2}\mathbf{B}_{cl} & \cdots \end{bmatrix} \|_{\infty/2}$$
$$\leq \| \begin{bmatrix} \mathbf{C}_{cl,i}\mathbf{A}_{cl}^{N+1}\mathbf{B}_{cl} & \mathbf{C}_{cl,i}\mathbf{A}_{cl}^{N+2}\mathbf{B}_{cl} & \cdots \end{bmatrix} \|_\infty \tag{34}$$

where $\|\cdot\|_{\infty/2}$ implies the induced norm from l_2 to l_∞. Here, the left-hand-side (LHS) of (34) can be represented by

$$\| \begin{bmatrix} \mathbf{C}_{cl,i}\mathbf{A}_{cl}^{N+1}\mathbf{B}_{cl} & \mathbf{C}_{cl,i}\mathbf{A}_{cl}^{N+2}\mathbf{B}_{cl} & \cdots \end{bmatrix} \|_{\infty/2}$$
$$= \sup_{\|\mathbf{w}\|_2 \leq 1} \| \begin{bmatrix} \mathbf{C}_{cl,i}\mathbf{A}_{cl}^{N+1}\mathbf{B}_{cl}\mathbf{w}_0 & \mathbf{C}_{cl,i}\mathbf{A}_{cl}^{N+2}\mathbf{B}_{cl}\mathbf{w}_1 & \cdots \end{bmatrix} \|_\infty$$
$$= \left(\sum_{k=N+1}^{\infty} (\mathbf{C}_{cl,i}\mathbf{A}_{cl}^k\mathbf{B}_{cl})(\mathbf{C}_{cl,i}\mathbf{A}_{cl}^k\mathbf{B}_{cl})^T \right)^{1/2}$$
$$= (\mathbf{C}_{cl,i}\mathbf{A}_{cl}^{N+1}X_{cl}(\mathbf{A}_{cl}^T)^{N+1}\mathbf{C}_{cl,i}^T)^{1/2} = \beta_{N,i}, \tag{35}$$

because the Cauchy inequality leads to

$$\left(\sum_{k=0}^{\infty} \mathbf{C}_{cl,i}\mathbf{A}_{cl}^{N+k+1}\mathbf{B}_{cl}\mathbf{w}_k \right)^2$$
$$\leq \sum_{k=0}^{\infty} (\mathbf{C}_{cl,i}\mathbf{A}_{cl}^{N+k+1}\mathbf{B}_{cl})(\mathbf{C}_{cl,i}\mathbf{A}_{cl}^{N+k+1}\mathbf{B}_{cl})^T \cdot \sum_{k=0}^{\infty} \mathbf{w}_k^T\mathbf{w}_k.$$

Combining (35) and (33) completes the proof.

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