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# Computing the $L_\infty$-Induced Norm of LTI Systems: Generalization of Piecewise Quadratic and Cubic Approximations

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**ABSTRACT** This paper is concerned with performance analysis for bounded persistent disturbances of continuous-time linear time-invariant (LTI) systems. Such an analysis can be done by computing the $L_\infty$-induced norm of continuous-time LTI systems since it corresponds to the worst maximum magnitude of the output for the worst persistent external input with a unit magnitude. In our preceding study, piecewise constant and linear approximation schemes for analyzing this norm have been developed through two alternative approximation approaches, one for the input and the other for the relevant kernel function, via the fast-lifting technique. The approximation errors in these approximation schemes have been shown to converge to 0 at the rates of $1/N$ and $1/N^2$, respectively, as the fast-lifting parameter $N$ is increased. Along this line, this paper aims at developing generalized techniques that offer improved accuracy named the piecewise quadratic and cubic approximation schemes. The generalization and the associated accuracy improvement discussed in this paper are not limited to the increased orders of approximation but are extended further to taking advantage of the freedom in the point around which relevant functions are expanded to Taylor series. The approximation errors in the piecewise quadratic and cubic approximation schemes are shown to converge to 0 at the rates of $1/N^3$ and $1/N^4$, respectively, regardless of the point at which the Taylor expansion is applied. Finally, effectiveness of the developed computation methods is confirmed through a numerical example.

**INDEX TERMS** Approximate computing, approximation methods, linear systems, performance analysis, robustness.

## I. INTRODUCTION

Mathematical models of the real control systems such as electrical circuit systems, mechanical systems and electromechanical systems are often described as continuous-time linear time-invariant (LTI) systems. On the other hand, because they are usually affected by unknown disturbances at either their input or output, evaluating the effect of such disturbances on the real control systems when they are under feedback control is quite important in the associated performance analysis problem. To put it another way, unexpected elements such as external disturbances are unavoidable in real control systems, and thus there have been a number of studies on evaluating performances of control systems against unknown disturbances. One of the most typical methods for evaluating such performances is to compute a system norm, which is adequately defined depending on the nature of the disturbances affecting the system and the desired performance specifications. Developing computation methods of system norms have been important issues in the field of control engineering.

Among various system norms used in the performance analysis for external disturbances, the $H_2$ and $H_\infty$ norms have been frequently considered, as in [1]–[7] and the references therein. More precisely, the studies on the $H_2$ norm take the $L_2$ norms of the outputs for impulse disturbance inputs while those on the $H_\infty$ norm consider the $L_2$ norm of the output for the worst disturbance inputs among those with

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a unit $L_2$ norm. Simply put, if the impulse response of a single-input/single-output (SISO) system is denoted by $g(t)$, the $H_2$ norm of this system is given by

$$\|g\|_{L_2} := \left(\int_0^\infty g^2(t)dt\right)^{1/2}$$

It is also well-known that the $H_2$ norm is closely related to the power of the output of this system under a white noise input. On the other hand, if a SISO system has the input $u(t)$ and the output $y(t)$, its $H_\infty$ norm is given by

$$\sup_{\|u\|_{L_2}\leq 1} \|y\|_{L_2}$$

It is obvious from the above equation that the $H_\infty$ norm corresponds to the worst energy of the output. To put it another way, because the $L_2$ norm of a signal corresponds to the energy of the signal, the studies on the $H_2$ and $H_\infty$ norms could be interpreted as taking the energy of the output as the performance measure.

However, the use of the $H_2$ and $H_\infty$ norms cannot suitably match the control applications such as collision avoidance of mechanical manipulators, protecting chemical plants from being overly pressured, and so on. This is because the maximum magnitude rather than the energy of the outputs should be evaluated in such control application problems. Furthermore, these norms do not correspond to dealing with bounded persistent disturbances, which are often encountered in real control systems because of unknown external elements such as unexpected environmental changes. In connection with this, practical issues for bounded persistent disturbances have been extensively discussed in various fields, such as robot manipulators [8], humanoids [9], aerospace systems [10], [11], model predictive control [12], Markovian jump systems [13], networked control systems [14], and so on. When we are in a position to consider the effect of bounded persistent disturbances, taking the $L_\infty$ norm is quite appropriate because the $L_\infty$ norm of a signal corresponds to its maximum amplitude and the $L_\infty$-induced norm of a system corresponds to the worst maximum magnitude of the output for bounded persistent disturbances with a unit magnitude. In this sense, this paper aims at numerically analyzing the $L_\infty$-induced norm of continuous-time finite-dimensional linear time-invariant (FDLTI) systems as accurately as possible (where the term of 'finite-dimensional' means that the size of state vector has a finite value).

### A. RELATED STUDIES ON THE $L_\infty$-INDUCED NORM ANALYSIS

The studies associated with the treatment of the $L_\infty$-induced norm have been called the $L_1$ problem because this induced norm is known to coincide with the $L_1$ norm of the impulse response of the system in the single-input/single-output (SISO) LTI case. The $L_1$ problem with a special case of SISO LTI systems is formulated in [15] for the first time, while a general case of SISO LTI systems is dealt with in [16], [17]. Subsequently, to alleviate the difficulties in computing the $L_1$ norm of the impulse response, an approximate computation method for the SISO LTI case is developed in [18]. More precisely, an integer parameter $N$ is introduced to divide a sufficiently large time interval $[0, H)$ into $N$ subintervals, on which the Cauchy-Schwarz inequality is applied to all the integrals of the absolute value of the impulse response. Such a procedure could lead to an algorithm, by which an upper bound and lower bound on the $L_1$ norm of the impulse response can be derived, and the gap between the bounds is suppressed to a pre-specified tolerance by taking a sufficiently large $N$. However, neither its extension to multi-input/ multi-output (MIMO) LTI systems nor the associated mathematical analysis on the convergence order about the parameter $N$ has been discussed in that study.

In contrast to the conventional studies [15]–[18], new methods named the input approximation approach [19] and the kernel approximation approach [20] have been recently introduced to tackle the $L_1$ analysis problem for the MIMO case directly. Both (i.e., the input and kernel) approximation approaches also take a sufficiently large time interval $[0, H)$ over which the associated input-output behavior is dealt with and some sorts of approximation schemes are developed in a piecewise manner. Namely, two types of schemes called the piecewise constant approximation scheme and the piecewise linear approximation scheme were studied for both fundamentally different approaches; the input approximation approach [19] deals with the input-output behavior through the approximation of the (external) input and output signals themselves, while the kernel approximation approach [20] applies approximation to the "internal equation" governing the input-output relation. More precisely, both approximation approaches consider specific approximations in an operator-theoretic fashion, where the fast-lifting treatment [21] plays an important role; its idea is to take an integer parameter $N$ and divide the interval $[0, H)$ into $N$ subintervals with an equal width without introducing sampling, and thus keeping the full information on the original functions on each subinterval allows us somewhat sophisticated approximations based on the Taylor expansion (with order 0 and 1) of relevant functions.

With the strength of the fast-lifting technique mentioned above, computable upper and lower bounds on the $L_\infty$-induced norm for the MIMO LTI case have been successfully derived through these approximation approaches. More importantly, it was shown in [19] and [20], both for the input approximation approach and the kernel approximation approach, respectively, that the gap between the derived upper and lower bounds converges to 0 at the rate of $1/N$ for the piecewise constant approximation scheme and at the rate of $1/N^2$ for the piecewise linear approximation scheme. Even though this might sound that the accuracy are the same for the input approximation and kernel approximation approaches as long as the same approximation scheme (or approximation order) is adopted, this is true only for a simplified view from the convergence rates, and a more important consequence of the study [20] from a numerically more quantitative point of view is that the kernel approximation approach leads to a smaller gap between upper and lower bounds.

### B. CONTRIBUTIONS AND ORGANIZATION OF THIS PAPER

Stimulated by the success of the $L_\infty$-induced norm analysis in the preceding studies, this paper pursues extended schemes named the piecewise quadratic and piecewise cubic approximations for achieving better convergence rates than those in [19], [20]; these new schemes are developed again under both the input and kernel approximation approaches through the fast-lifting treatment together with the arguments of the Taylor expansion of relevant functions.

These schemes readily allow us to compute upper and lower bounds on the $L_\infty$-induced norm of MIMO LTI systems, and the gap between the upper and lower bounds is shown to converge to 0 at the improved rates of $1/N^3$ and $1/N^4$ in the piecewise quadratic and piecewise cubic approximation schemes, respectively, regardless of the input approximation approach or the kernel approximation approach. Very importantly, the improvement over our earlier studies [19], [20] is not limited simply to the use of such higher order approximation schemes (and establishing theoretical bases for these schemes) but includes generalized arguments relevant to how the Taylor expansion of relevant functions is used; even though the earlier studies only considered the Taylor expansion at the beginning of each subintervals resulting from the application of fast-lifting, this paper considers taking advantage of the freedom in the time instant around which relevant functions are expanded to Taylor series. In this respect, this paper corresponds to a significantly extended version of an earlier conference paper by the authors [22], which discussed the development of the piecewise quadratic and cubic approximation schemes for the first time (without associated proofs and numerical examples). Under such generalized treatment, it is once again shown that the kernel approximation approach is quantitatively superior to the input approximation approach in terms of the gap between the upper and lower bounds, even though the convergence rate itself (as mentioned above) is shared by the two approaches under the same approximation order.

The organization of this paper is as follows. In Section II, we first state our problem and describe some preliminary arguments for our input and kernel approximation approaches through piecewise approximation schemes. The input approximation approach is considered in Section III while the kernel approximation approach is dealt with in Section IV. Even though these two sections are confined to treatment over a sufficiently large but finite time interval, Section V provides unified arguments about the ultimate $L_\infty$-induced norm computation for the input approximation and kernel approximation approaches with the treatment over the infinite time interval $[0,\infty)$. In Section VI, a numerical example is provided to demonstrate the effectiveness of the computation approaches introduced in this paper. Finally, we state concluding remarks in Section VII.

Throughout the paper, we use the following notations. The symbols $\mathbb{R}^\nu$ and $\mathbb{N}$ are used to denote the sets of $\nu$-dimensional real numbers and positive integers, respectively, while $\mathbb{N}_0$ is used to imply $\mathbb{N}\cup\{0\}$. The notation $|\cdot|_\infty$ is used to mean either the $\infty$-norm of a finite-dimensional vector, i.e.,

$$|v|_\infty := \max_{1\le i\le \nu} |v_i| \tag{1}$$

for $v\in\mathbb{R}^\nu$, where $v_i$ is the $i$th element of $v$, or the $\infty$-norm of a finite-dimensional matrix, i.e.,

$$|A|_\infty := \sup_{|v|_\infty\le 1} \frac{|Av|_\infty}{|v|_\infty} = \max_{1\le i\le \nu_1} \sum_{j=1}^{\nu_2} |A_{ij}| \tag{2}$$

for $A\in\mathbb{R}^{\nu_1\times\nu_2}$, where $A_{ij}$ is the $(i,j)$ element of $A$. The notation $\|\cdot\|_\infty$ is used to mean either of the following: the $L_\infty[0,H)$ norm of a vector function, i.e.,

$$\|f(\cdot)\|_\infty := \operatorname*{ess\,sup}_{0\le t<H} |f(t)|_\infty \tag{3}$$

that with $H$ replaced by $H/N(=:h)$ or $\infty$, the $L_\infty[0,H)$-induced norm of an operator, i.e.,

$$\|\mathcal{F}\|_\infty := \sup_{\|w\|_\infty\le 1} \frac{\|\mathcal{F}w\|_\infty}{\|w\|_\infty} \tag{4}$$

or that with $H/N(=:h)$ or $\infty$ instead of $H$, whose distinction will be clear from the context.

To summarize, the $L_\infty$ norm of a signal and the $L_\infty$-induced norm of a system are defined as (3) and (4), respectively, with taking $H\to\infty$. Because these two norms are based on the vector $\infty$-norm defined as (1), it is obvious from (3) that the $L_\infty$ norm is dependent on the dimension of input or output signal, and thus the $L_\infty$-induced norm formulated by using the $L_\infty$ norm as shown in (4) is also dependent on the dimensions of both input and output signals.

## II. PRELIMINARIES FOR COMPUTING THE $L_\infty$-INDUCED NORM

Let us first consider definition of the $L_\infty$ space. For a vector-valued function $f(\cdot)$, we say that $f(\cdot)$ is an element of $L_\infty$ if its $L_\infty$ norm given by (3) is well-defined and bounded. The elements of $L_\infty$ are usually called bounded persistent signals. We next deal with the stable continuous-time linear time-invariant (LTI) system described by

$$\begin{cases} \dot{x} = Ax + Bw \\ z = Cx + Dw \end{cases} \tag{5}$$

where $x(t)\in\mathbb{R}^n$ is the state, $w(t)\in\mathbb{R}^{n_w}$ is the input and $z(t)\in\mathbb{R}^{n_z}$ is the output. For the continuous-time LTI system given by (5), this paper considers the worst maximum magnitude of $z$ for bounded persistent $w$ as a performance measure. In other words, $w$ and $z$ are assumed to belong to $(L_\infty)^{n_w}$ and $(L_\infty)^{n_z}$, respectively, and this paper aims at computing the $L_\infty$-induced norm from $w$ to $z$ as shown in Fig 1.

To this end, we first note the input/output relation of (5) through the convolution integral

$$z(t) = \int_0^t C\exp(A(t-\theta))Bu(\theta)d\theta + Du(t) \tag{6}$$

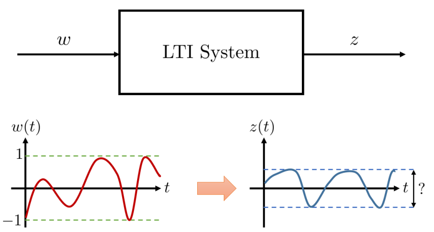


**FIGURE 1.** The problem of $L_\infty$-induced norm analysis.

and define the operator $\mathcal{G} : (L_\infty)^{n_w} \to (L_\infty)^{n_z}$ associated with the above convolution integral by $(\mathcal{G}w)(t) = z(t)$.

Regarding a sophisticated computation of the $L_\infty$-induced norm $\|\mathcal{G}\|_\infty$, we introduce the following alternative representation [20] with $u(\cdot) := w(t-\cdot)$ to alleviate the difficulty about the convolution integral in computing $\|\mathcal{G}\|_\infty$:

$$\begin{aligned}\|\mathcal{G}\|_\infty &= \sup_{0\le t<\infty} \sup_{\|u\|_\infty\le 1} \left| \int_0^t C\exp(A\theta)Bu(\theta) + Du(0) \right|_\infty \\ &=: \sup_{0\le t<\infty} \sup_{\|u\|_\infty\le 1} |(\mathcal{F}u)(t)|_\infty \\ &= \lim_{t\to\infty} \sup_{\|u\|_\infty\le 1} |(\mathcal{F}u)(t)|_\infty = \|\mathcal{F}\|_\infty \end{aligned} \quad (7)$$

### A. TRUNCATION TREATMENT OF OPERATOR $\mathcal{F}$

We take sufficiently large $H$ to confine our attention to the interval $[0, H)$ in the treatment of $\mathcal{F}$ operating over $[0, \infty)$ so that the difficulty in the computation of $\|\mathcal{F}\|_\infty$ would be alleviated. We then first note for $t \ge H$ that

$$(\mathcal{F}u)(t) = \mathcal{F}_H^- u + (\mathcal{F}_H^+ u)(t) \quad (8)$$

where

$$\mathcal{F}_H^- u := \int_0^H C\exp(A\theta)Bu(\theta)d\theta + Du(0) \quad (9)$$

$$(\mathcal{F}_H^+ u)(t) := \int_H^t C\exp(A\theta)Bu(\theta)d\theta \quad (10)$$

In view of (7), $\|\mathcal{F}\|_\infty$ is bounded by the inequality

$$\|\mathcal{F}_H^-\|_\infty \le \|\mathcal{F}\|_\infty \le \|\mathcal{F}_H^-\|_\infty + \|\mathcal{F}_H^+\|_\infty \quad (11)$$

where

$$\|\mathcal{F}_H^-\|_\infty := \sup_{\|u\|_\infty\le 1} |\mathcal{F}_H^- u|_\infty \quad (12)$$

$$\|\mathcal{F}_H^+\|_\infty := \lim_{t\to\infty} \sup_{\|u\|_\infty\le 1} |(\mathcal{F}_H^+ u)(t)|_\infty \quad (13)$$

Here, it would be worthwhile to remark that the stability assumption of (5) ensures that $\|\mathcal{F}_H^+\|_\infty \to 0$ as $H \to \infty$ (see (32) below taken from [20]). In this sense, our basic strategy for computing $\|\mathcal{F}\|_\infty$ is to derive upper and lower bounds of $\|\mathcal{F}_H^-\|_\infty$ as accurately as possible through approximation arguments improved over the preceding studies [19], [20] (and the associated error bound analysis) while $\|\mathcal{F}_H^+\|_\infty$ is treated in a comparatively simple way by deriving its upper bound through the arguments similar to [20].

*Remark 1:* The preceding studies [19], [20] took $\|\mathcal{F}_H^-\|_\infty - \|\mathcal{F}_H^+\|_\infty$ as a lower bound in (11). The above improvement is straightforward but is first provided in the present paper.

### B. FAST-LIFTING APPROACH TO $\mathcal{F}_H^-$

This subsection reviews the fast-lifting treatment [21] with the parameter $N \in \mathbb{N}$ as the first step to develop approximation methods of $\mathcal{F}_H^-$. For $h := H/N$, we introduce the mapping from $u \in (L_\infty[0, H))^{n_w}$ to $\check{u} := [(u^{(1)})^T \ \cdots \ (u^{(N)})^T]^T \in (L_\infty[0, h))^{n_w N}$ denoted by $\check{u} = \mathbf{L}_N u$ [21], where

$$u^{(m)}(\theta) = u((m-1)h + \theta) \quad (0 \le \theta < h) \quad (14)$$

It immediately follows from (9) that

$$\begin{aligned}\mathcal{F}_H^- u &= Du^{(1)}(0) + \sum_{p=1}^{N} C(A_d)^{p-1}\mathbf{B}u^{(m)} \\ &= Du^{(1)}(0) + C_{dN}\overline{\mathbf{B}}\check{u} =: \mathcal{F}_{HN}^- \check{u}\end{aligned} \quad (15)$$

where

$$A_d := \exp(Ah), \quad \mathbf{B}u^{(m)} := \int_0^h \exp(A\theta)Bu^{(m)}(\theta)d\theta \quad (16)$$

$$C_{dN} := \begin{bmatrix} C & CA_d & \cdots & C(A_d)^{N-1} \end{bmatrix} \quad (17)$$

and $\overline{(\cdot)}$ denotes $diag[(\cdot), \cdots, (\cdot)]$ consisting of $N$ copies of $(\cdot)$. We can see from the definition of $\mathbf{L}_N$ that $\mathcal{F}_{HN}^- = \mathcal{F}_H^- \mathbf{L}_N^{-1}$ and $\|\mathcal{F}_H^-\|_\infty = \|\mathcal{F}_{HN}^-\|_\infty$. Hence, dealing with $\mathcal{F}_{HN}^-$ instead of $\mathcal{F}_H^-$ is convenient since the input or kernel function of $\mathbf{B}$ involved in the former operator (i.e., $\mathcal{F}_{HN}^-$) is confined to the smaller interval $[0, h)$ (rather than $[0, H)$ on which $\mathcal{F}_H^-$ is defined) and the treatment of $D$ is essentially the same for both operators. This gives us a better chance to derive sophisticated piecewise approximation arguments over the interval $[0, H)$. While our preceding studies [19], [20] provided the piecewise constant and linear approximation schemes through this idea, this paper aims at developing higher-order approximation schemes. More precisely, we approximate the input or kernel function of $\mathbf{B}$ by the second- or third-order polynomials and develop what we call the piecewise quadratic approximation scheme and the piecewise cubic approximation scheme. All these methods can eventually be interpreted as different approximations of the Taylor expansion of the kernel function of $\mathbf{B}$ defined on $[0, h)$.

## III. INPUT APPROXIMATION APPROACH

This section deals with the input approximation approach to $\mathcal{F}_H^-$, where the quadratic and cubic approximation schemes of the input of $\mathbf{B}$ on $[0, h)$ are developed in a quite nontrivial way (by which we can lead to the piecewise quadratic and cubic approximation schemes of $\mathcal{F}_H^-$ on $[0, H)$). The associated convergence rates are further shown to be in the orders of $1/N^3$ and $1/N^4$, respectively.

### A. PIECEWISE QUADRATIC AND CUBIC APPROXIMATIONS

The basic idea of the piecewise $l$th order input approximation approach with $l \in \mathbb{N}_0$ is to introduce the operator $\mathbf{J}_l^{[\alpha]} : (L_\infty[0, h))^{n_w} \to (L_\infty[0, h))^{n_w}$ ($\alpha \in [0, 1]$) to approximate the input $u$ over the interval $[0, h)$ with the $l$th order polynomial $\mathbf{J}_l^{[\alpha]}u$, where

$$(\mathbf{J}_l^{[\alpha]}u)(\theta) = \sum_{p=0}^{l}(\theta - \alpha h)^p \int_0^h f_{lp}^{[\alpha]}(\tau)u(\tau)d\tau \quad (0 \le \theta < h) \tag{18}$$

with adequately defined scalar-valued functions $f_{lp}^{[\alpha]}(\tau)$ $(p = 0, \ldots, l)$. Once $\alpha \in [0, 1]$ and these functions are given, they determine $l$th order polynomial approximation of $u \in (L_\infty[0, h))^{n_w}$. This can alternatively be interpreted as approximating $\mathbf{B}$ by $\mathbf{B}_{il}^{[\alpha]} := \mathbf{B} \cdot \mathbf{J}_l^{[\alpha]}$, where the subscript $i$ stands for the input approximation approach. This further leads to approximating $\mathcal{F}_{HN}^{-}$ by replacing $\mathbf{B}$ with $\mathbf{B}_{il}^{[\alpha]}$ in (15), i.e., by

$$\mathcal{F}_{HNil}^{-[\alpha]}\check{u} := Du^{(1)}(0) + C_{dN}\overline{\mathbf{B}_{il}^{[\alpha]}}\check{u} \tag{19}$$

As shown in the appendix, this induces approximation of the kernel function $\exp(A\tau)B$ of $\mathbf{B}$ by the $l$th order polynomial $\sum_{p=0}^{l} f_{lp}^{[\alpha]}(\tau)B_{pd}^{[\alpha]}$, where

$$B_{pd}^{[\alpha]} := \int_0^h \exp(A\theta)(\theta - \alpha h)^p B d\theta \quad (p = 0, \cdots, l) \tag{20}$$

In connection with this, our preceding study [20] has dealt with the piecewise constant and linear approximations (i.e., $l = 0$ and $l = 1$) only for $\alpha = 0$ and succeeded in deriving the functions $f_{lp}^{[0]}(\tau)$ $(p = 0, \ldots, l)$ 'appropriately' in the sense that they successfully lead to the convergence rates of the approximation errors of $\|\mathcal{F}_{HN}^{-}\|_\infty$ by $\|\mathcal{F}_{HNil}^{-[\alpha]}\|_\infty$ in the orders of $1/N^{l+1}$ $(l = 0, 1)$. Extending the arguments for determining the functions $f_{lp}^{[\alpha]}(\tau)$ to the case of the piecewise quadratic approximation scheme $(l = 2)$ and general $\alpha$, we are reasonably led to the use of the functions $f_{2p}^{[\alpha]}(\tau)$ in the form of the quadratic functions

$$f_{2p}^{[\alpha]}(\tau) = \sum_{j=0}^{2} \frac{c_{2pj}^{[\alpha]}}{h^{p+j+1}}(\tau - \alpha h)^j \quad (p = 0, 1, 2) \tag{21}$$

where the coefficients $c_{2pj}^{[\alpha]}$ $(p, j = 0, 1, 2)$ are independent of $h$. Then, we can determine these coefficients in such a way that a portion of the function $\sum_{p=0}^{l} f_{lp}^{[\alpha]}(\tau)B_{pd}^{[\alpha]}$ (almost) matches the Taylor expansion (around $\tau = \alpha h$) of the kernel function $\exp(A\tau)B$ of $\mathbf{B}$ up to the second order in $\tau$, by which the $L_1[0, h)$ norm of each entry of the remaining portion is made to be bounded in the order of $h^4$; this, in turn, makes the convergence rate of the approximation error of $\|\mathcal{F}_{H}^{-}\|_\infty = \|\mathcal{F}_{HN}^{-}\|_\infty$ by $\|\mathcal{F}_{HNi2}^{-[\alpha]}\|_\infty$ (where the latter can be computed explicitly) in the order of $h^3$ and thus $1/N^3$ as $N \to \infty$ (as we show in the following subsection). Through such arguments, we arrive at the coefficients given in Table 1, whose detailed derivation process is given in the appendix.

**TABLE 1.** Coefficients in piecewise quadratic approximation scheme.

| | |
|---|---|
| $c_{200}^{[\alpha]}$ | $9(20\alpha^4 - 40\alpha^3 + 28\alpha^2 - 8\alpha + 1)$ |
| $c_{201}^{[\alpha]}$ | $36(10\alpha^3 - 15\alpha^2 + 7\alpha - 1)$ |
| $c_{202}^{[\alpha]}$ | $30(6\alpha^2 - 6\alpha + 1)$ |
| $c_{210}^{[\alpha]}$ | $36(10\alpha^3 - 15\alpha^2 + 7\alpha - 1)$ |
| $c_{211}^{[\alpha]}$ | $48(15\alpha^2 - 15\alpha + 4)$ |
| $c_{212}^{[\alpha]}$ | $180(2\alpha - 1)$ |
| $c_{220}^{[\alpha]}$ | $30(6\alpha^2 - 6\alpha + 1)$ |
| $c_{221}^{[\alpha]}$ | $180(2\alpha - 1)$ |
| $c_{222}^{[\alpha]}$ | $180$ |

Similarly for the case of the piecewise cubic approximation scheme $(l = 3)$, we are led to

$$f_{3p}^{[\alpha]}(\tau) = \sum_{j=0}^{3} \frac{c_{3pj}^{[\alpha]}}{h^{p+j+1}}(\tau - \alpha h)^j \quad (p = 0, 1, 2, 3) \tag{22}$$

with $c_{3pj}^{[\alpha]}$ $(p, j = 0, 1, 2, 3)$ independent of $h$ and given in Table 2 (provided that these functions are assumed to be cubic functions), which are also obtained through the third-order truncation of the Taylor expansion (around $\tau = \alpha h$) of the kernel function $\exp(A\tau)B$ of $\mathbf{B}$ and allow us to obtain the improved convergence order of $1/N^4$ for the approximation error of $\|\mathcal{F}_{H}^{-}\|_\infty = \|\mathcal{F}_{HN}^{-}\|_\infty$ by $\|\mathcal{F}_{HNi3}^{-[\alpha]}\|_\infty$ (where the latter can be computed explicitly) as we show in the following subsection, compared with that for the piecewise quadratic approximation scheme given by $1/N^3$.

### B. CONVERGENCE AND upper/LOWER BOUND ANALYSIS

In connection with the approximation of input by the above piecewise quadratic approximation and piecewise cubic approximation schemes, we are led to the following two lemmas.

*Lemma 1 (Piecewise Quadratic/Cubic Approximation):* For $l = 2$ and 3, let $T_{lm}^{[\alpha]}$ $(m = 1, \ldots, N)$ be the matrix consisting of the $L_1[0, h)$ norm of each entry of the matrix quadratic/cubic function

$$C(A_d)^{m-1}\sum_{p=0}^{l} G_{lp}^{[\alpha]}(\theta - \alpha h)^p \tag{23}$$

where the matrices $G_{lp}^{[\alpha]}$ $(p = 0, \cdots, l)$ are defined as

$$G_{lp}^{[\alpha]} := \sum_{j=0}^{l} B_{jd}^{[\alpha]} \frac{c_{ljp}^{[\alpha]}}{h^{p+j+1}} \tag{24}$$

Then, we have $\|\mathcal{F}_{HNil}^{-[\alpha]}\|_\infty = |F_{HNil}^{-[\alpha]}|_\infty$ with

$$F_{HNil}^{-[\alpha]} := \begin{bmatrix} D & T_{l1}^{[\alpha]} & \cdots & T_{lN}^{[\alpha]} \end{bmatrix} \tag{25}$$

*Lemma 2:* The inequality

$$\|\mathcal{F}_{HN}^{-} - \mathcal{F}_{HNil}^{-[\alpha]}\|_\infty \le \frac{K_{Nil}^{[\alpha]}}{N^{l+1}} \quad (l = 2, 3) \tag{26}$$

**TABLE 2. Coefficients in piecewise cubic approximation scheme.**

| | |
|---|---|
| $c_{300}^{[\alpha]}$ | $16(175\alpha^6 - 525\alpha^5 + 615\alpha^4 - 355\alpha^3 + 105\alpha^2 - 15\alpha + 1)$ |
| $c_{301}^{[\alpha]}$ | $120(70\alpha^5 - 175\alpha^4 + 164\alpha^3 - 71\alpha^2 + 14\alpha - 1)$ |
| $c_{302}^{[\alpha]}$ | $240(35\alpha^4 - 70\alpha^3 + 48\alpha^2 - 13\alpha + 1)$ |
| $c_{303}^{[\alpha]}$ | $140(20\alpha^3 - 30\alpha^2 + 12\alpha - 1)$ |
| $c_{310}^{[\alpha]}$ | $120(70\alpha^5 - 175\alpha^4 + 164\alpha^3 - 71\alpha^2 + 14\alpha - 1)$ |
| $c_{311}^{[\alpha]}$ | $1200(21\alpha^4 - 42\alpha^3 + 30\alpha^2 - 9\alpha + 1)$ |
| $c_{312}^{[\alpha]}$ | $900(28\alpha^3 - 42\alpha^2 + 20\alpha - 3)$ |
| $c_{313}^{[\alpha]}$ | $1680(5\alpha^2 - 5\alpha + 1)$ |
| $c_{320}^{[\alpha]}$ | $240(35\alpha^4 - 70\alpha^3 + 48\alpha^2 - 13\alpha + 1)$ |
| $c_{321}^{[\alpha]}$ | $900(28\alpha^3 - 42\alpha^2 + 20\alpha - 3)$ |
| $c_{322}^{[\alpha]}$ | $720(35\alpha^2 - 35\alpha + 9)$ |
| $c_{323}^{[\alpha]}$ | $4200(2\alpha - 1)$ |
| $c_{330}^{[\alpha]}$ | $140(20\alpha^3 - 30\alpha^2 + 12\alpha - 1)$ |
| $c_{331}^{[\alpha]}$ | $1680(5\alpha^2 - 5\alpha + 1)$ |
| $c_{332}^{[\alpha]}$ | $4200(2\alpha - 1)$ |
| $c_{333}^{[\alpha]}$ | $2800$ |

holds, where $K_{Nil}^{[\alpha]}$ is defined as

$$K_{Nil}^{[\alpha]} := \frac{|C_{dN}|_\infty}{N} \cdot |\exp(A\alpha h)A^{l+1}B|_\infty \cdot e^{|A|_\infty h} \times H^{l+2} \cdot K_{il}^{[\alpha]} \tag{27}$$

with $K_{il}^{[\alpha]}$ given by

$$K_{il}^{[\alpha]} := \sum_{p=0}^{l}\sum_{j=0}^{l}\Big(\frac{(1-\alpha)^{l+p+2}+\alpha^{l+p+2}}{(l+1)!\cdot(l+p+2)} \times \frac{(1-\alpha)^{j+1}+\alpha^{j+1}}{j+1}\cdot|c_{lpj}^{[\alpha]}|\Big) + \frac{(1-\alpha)^{l+2}+\alpha^{l+2}}{(l+2)!} \tag{28}$$

Furthermore, $K_{Nil}^{[\alpha]}$ has the uniform upper bound with respect to $N$ given by

$$K_{ilU}^{[\alpha]} := |C|_\infty \cdot |A^{l+1}B|_\infty \cdot e^{(1+\alpha)\cdot H\cdot|A|_\infty} \cdot H^{l+2} \cdot K_{il}^{[\alpha]} \tag{29}$$

*Remark 2:* The assertions of the above lemmas corresponding to $l = 0, 1$ with $\alpha = 0$ have been established in [19], [20] for piecewise constant and linear approximations, and possibly hold also for even higher-order approximations. However, such a direction does not necessarily seem to be appealing as stated in a concluding remark in Section VII.

*Remark 3:* $K_{il}^{[\alpha]}$ of (28) corresponds to the $L_1[0,h)$ norm of the matrix $\infty$-norm for the difference between $\sum_{p=0}^{l} f_{lp}^{[\alpha]}(\tau)B_{pd}^{[\alpha]}$ and $\exp(A\tau)B$, and it is derived by using the Taylor expansions of $\exp(A\tau)$ around $\tau = \alpha h$ appearing in the former and the definition of the latter (i.e., $\exp(A\theta)$ in (16)). In connection with this, roughly speaking, the first term in (28) is related with the $L_1[0,h)$ norm of the matrix $\infty$-norm for the trailing part of the Taylor expansion of the former with the orders in $\tau$ higher than $l$, while the second term in (28) corresponds to that of the latter with the orders in $\theta$ higher than $l$. In contrast, $K_{kl}^{[\alpha]}$, which will be defined in the following section relevant to the kernel approximation approach, coincides with the second term in (28) since the kernel approximation approach completely matches the $l$th order truncation of the Taylor expansion of $\exp(A\tau)B$ around $\tau = \alpha h$ (see (37) for details).

The aim of Lemma 1 is to provide the explicit methods for computing $\|\mathcal{F}_{HNil}^{-[\alpha]}\|_\infty$ $(l = 2, 3)$ approximating $\|\mathcal{F}_H^-\|_\infty$ $(= \|\mathcal{F}_{HN}^-\|_\infty)$, while that of Lemma 2 is to give the associated error bounds between $\|\mathcal{F}_{HNil}^{-[\alpha]}\|_\infty$ $(l = 2, 3)$ and $\|\mathcal{F}_H^-\|_\infty (= \|\mathcal{F}_{HN}^-\|_\infty)$. The proofs of these lemmas are given in the appendix since they are quite technical. Combining Lemmas 1 and 2 together with $\|\mathcal{F}_{HN}^-\|_\infty = \|\mathcal{F}_H^-\|_\infty$ leads to the following theorem.

*Theorem 1:* The following inequality holds for $l = 2, 3$:

$$|F_{HNil}^{-[\alpha]}|_\infty - \frac{K_{Nil}^{[\alpha]}}{N^{l+1}} \le \|\mathcal{F}_H^-\|_\infty \le |F_{HNil}^{-[\alpha]}|_\infty + \frac{K_{Nil}^{[\alpha]}}{N^{l+1}} \tag{30}$$

Theorem 1 clearly means that the matrix $\infty$-norm $|F_{HNil}^{-[\alpha]}|_\infty$ tends to $\|\mathcal{F}_H^-\|_\infty$ at the convergence rate of $1/N^{l+1}$ as $N \to \infty$ (by the existence of the uniform upper bound $K_{ilU}^{[\alpha]}$ for $K_{Nil}^{[\alpha]}$). On the basis of the above theorem, the details of our ultimate main results relevant to the upper and lower bounds on the $L_\infty$-induced norm with $\|\mathcal{F}_H^+\|_\infty$ in (11) taken into account will be provided in Section V, including those for the kernel approximation approach discussed in the following section.

## IV. KERNEL APPROXIMATION APPROACH

This section provides the kernel approximation approach to $\mathcal{F}_H^-$, where the (direct as opposed to that induced by the input approximation approach discussed in the preceding section) quadratic and cubic approximations of the kernel function $\exp(A\theta)B$ of $\mathbf{B}$ on $[0, h)$ are introduced (by which we can achieve the piecewise quadratic and cubic approximation schemes of $\mathcal{F}_H^-$ on $[0, H)$). It is also shown that the associated convergence rates are in the orders of $1/N^3$ and $1/N^4$, respectively. These orders are the same as those for the input approximation approach in the preceding section, but we further discuss a more quantitative aspect of the advantage obtained by the kernel approximation approach.

### A. PIECEWISE QUADRATIC AND CUBIC APPROXIMATIONS

Let us introduce the operator $\mathbf{B}_{kl}^{[\alpha]} : (L_\infty[0, h))^{n_w} \to \mathbb{R}^n$ described by

$$\mathbf{B}_{kl}^{[\alpha]} u = \sum_{p=0}^{l} \int_0^h \exp(A\alpha h) \frac{(A(\theta - \alpha h))^p}{p!} B u(\theta) d\theta \quad (l = 2, 3;\ \alpha \in [0, 1]) \tag{31}$$

Here, the subscript k stands for the kernel approximation. Introducing this operator corresponds to the $l$th-order truncation of the Taylor expansion (around $\theta = \alpha h$) of the kernel function $\exp(A\theta)B = \sum_{p=0}^{\infty} \exp(A\alpha h) \frac{(A(\theta - \alpha h))^p}{p!}$ of the operator $\mathbf{B}$.

Next, consider the operator $\mathcal{F}_{HNkl}^{-[\alpha]}$ obtained by replacing $\mathbf{B}$ with $\mathbf{B}_{kl}^{[\alpha]}$ in (15), i.e.,

$$\mathcal{F}_{HNkl}^{-[\alpha]} \breve{u} := D u^{(1)}(0) + C_{dN} \overline{\mathbf{B}_{kl}^{[\alpha]}} \breve{u} \tag{32}$$

which is precisely the idea of kernel approximation. In the following subsection, it would be shown that $\|\mathcal{F}_{HNk2}^{-[\alpha]}\|_\infty$ and $\|\mathcal{F}_{HNk3}^{-[\alpha]}\|_\infty$ can be explicitly computed and tend to $\|\mathcal{F}_H^-\|_\infty$ as $N \to \infty$ with the associated convergence rates of $1/N^3$ and $1/N^4$, respectively.

### B. CONVERGENCE AND UPPER/LOWER BOUND ANALYSIS

With respect to the approximation of the kernel function by the above piecewise quadratic approximation and piecewise cubic approximation schemes, we give the following two lemmas.

*Lemma 3 (Piecewise Quadratic/Cubic Approximation):* For $l = 2$ and 3, let $Y_{lm}^{[\alpha]}$ $(m = 1, \ldots, N)$ be the matrix consisting of the $L_1[0, h)$ norm of each entry of the matrix quadratic/cubic function

$$C(A_d)^{m-1} \sum_{p=0}^{l} \exp(A\alpha h) \frac{(A(\theta - \alpha h))^p}{p!} \tag{33}$$

contained in (32). Then, $\|\mathcal{F}_{HNkl}^{-[\alpha]}\|_\infty$ coincides with the $\infty$-norm of the finite-dimensional matrix $F_{HNkl}^{-[\alpha]}$ defined as

$$F_{HNkl}^{-[\alpha]} := \begin{bmatrix} D & Y_{l1}^{[\alpha]} & \cdots & Y_{lN}^{[\alpha]} \end{bmatrix} \tag{34}$$

*Lemma 4:* The inequality

$$\|\mathcal{F}_{HN}^- - \mathcal{F}_{HNkl}^{-[\alpha]}\|_\infty \le \frac{K_{Nkl}^{[\alpha]}}{N^{l+1}} \quad (l = 2, 3) \tag{35}$$

holds, where $K_{Nkl}^{[\alpha]}$ is defined as

$$K_{Nkl}^{[\alpha]} := \frac{|C_{dN}|_\infty}{N} \cdot |\exp(A\alpha h) A^{l+1} B|_\infty \cdot e^{|A|_\infty h} \times H^{l+2} \cdot K_{kl}^{[\alpha]} \tag{36}$$

with $K_{kl}^{[\alpha]}$ given by

$$K_{kl}^{[\alpha]} := \frac{(1-\alpha)^{l+2} + \alpha^{l+2}}{(l+2)!} \tag{37}$$

Furthermore, $K_{Nkl}^{[\alpha]}$ has the following uniform upper bound with respect to $N$.

$$K_{klU}^{[\alpha]} := |C|_\infty \cdot |A^{l+1}B|_\infty \cdot e^{(1+\alpha) \cdot H \cdot |A|_\infty} \cdot H^{l+2} \cdot K_{kl}^{[\alpha]} \tag{38}$$

Lemma 3 corresponds to the exact computation method for $\|\mathcal{F}_{HNkl}^{-[\alpha]}\|_\infty$ $(l = 2, 3)$ approximating $\|\mathcal{F}_H^-\|_\infty$ (as well as $\|\mathcal{F}_{HN}^-\|_\infty$) while Lemma 4 provides the associated error bounds between $\|\mathcal{F}_{HNkl}^{-[\alpha]}\|_\infty$ $(l = 2, 3)$ and $\|\mathcal{F}_H^-\|_\infty$ $(= \|\mathcal{F}_{HN}^-\|_\infty)$. Their proofs are provided in Appendix VII because they are quite technical. By combining the above two lemmas, we are immediately led to the following theorem.

*Theorem 2:* The following inequality holds for $l = 2, 3$:

$$|F_{HNkl}^{-[\alpha]}|_\infty - \frac{K_{Nkl}^{[\alpha]}}{N^{l+1}} \le \|\mathcal{F}_H^-\|_\infty \le |F_{HNkl}^{-[\alpha]}|_\infty + \frac{K_{Nkl}^{[\alpha]}}{N^{l+1}} \tag{39}$$

*Remark 4:* In the same line of Remark 3, it readily follows from (27) and (36) that

$$K_{Nkl}^{[\alpha]} \le K_{Nil}^{[\alpha]} \tag{40}$$

This clearly implies that even though the kernel approximation and input approximation approaches share the same convergence rate when they take the approximation scheme of the same order $l$, the former is quantitatively superior to the latter when we consider the gap between the upper and lower bounds on $\|\mathcal{F}_H^-\|_\infty$.

## V. ULTIMATE COMPUTATION OF THE $L_\infty$-INDUCED NORM

This section provides ultimate methods for computing upper and lower bounds on $\|\mathcal{F}\|_\infty$ by using Theorems 1 and 2 together with the upper bound computation of $\|\mathcal{F}_H^+\|_\infty$ discussed in [20].

*Proposition 1 in [20]*: For $q > 0$ such that $|\exp(Aq)|_\infty < 1$, the inequality

$$\|\mathcal{F}_H^+\|_\infty \le \frac{q|B|_\infty \exp(|A|_\infty q)}{1 - |\exp(Aq)|_\infty} \cdot |C \exp(AH)|_\infty =: K_{Hq} \tag{41}$$

holds, and $K_{Hq}$ converges to 0 regardless of $q$ as $H \to \infty$.

Combining Theorems 1 and 2 together with Proposition 1 in [13] immediately leads to the following theorem, which is the ultimate main result of this paper.

*Theorem 3:* For $l = 2, 3$, $\alpha \in [0, 1]$ and $q > 0$ such that $|\exp(Aq)|_\infty < 1$, the following inequalities hold:

$$|F_{HNil}^{-[\alpha]}|_\infty - \frac{K_{Nil}^{[\alpha]}}{N^{l+1}} \le \|\mathcal{F}\|_\infty \le |F_{HNil}^{-[\alpha]}|_\infty + \frac{K_{Nil}^{[\alpha]}}{N^{l+1}} + K_{Hq} \quad (42)$$

$$|F_{HNkl}^{-[\alpha]}|_\infty - \frac{K_{Nkl}^{[\alpha]}}{N^{l+1}} \le \|\mathcal{F}\|_\infty \le |F_{HNkl}^{-[\alpha]}|_\infty + \frac{K_{Nkl}^{[\alpha]}}{N^{l+1}} + K_{Hq} \quad (43)$$

Furthermore, $K_{Nil}^{[\alpha]}$ and $K_{Nkl}^{[\alpha]}$ have uniform upper bounds $K_{ilU}^{[\alpha]}$ and $K_{klU}^{[\alpha]}$ in (29) and (38), respectively, while $K_{Hq}$ converges to 0 regardless of $q$ as $H \to \infty$.

Regarding a guideline to take the parameters $H$, $N$ and $q$, it is worthwhile to remark that the uniform upper bounds $K_{ilU}^{[\alpha]}$ and $K_{klU}^{[\alpha]}$ given in (29) and (38), respectively, are dependent on $H$ and increase as $H$ becomes larger to reduce $K_{Hq}$. However, because $K_{Hq}$ is bounded from above in the exponential order $e^{\delta H}$ regardless of $q$, where $\delta < 0$ is the maximum real part of the eigenvalues of $A$, the value of $K_{Hq}$ could be expected to be made small with a modest $H$. Hence, the uniform upper bounds $K_{ilU}^{[\alpha]}$ and $K_{klU}^{[\alpha]}$ can be kept modest. In this sense, it would be reasonable to take a relatively small $q$ as long as $|\exp(Aq)|_\infty < 1$ to avoid undue increase of $\exp(|A|_\infty q)$ in $K_{Hq}$. With a fixed $q$, the next step would be to take a sufficiently large $H$ such that $K_{Hq}$ is as small as we wish. Once $q$ and $H$ are fixed, the uniform upper bounds $K_{ilU}^{[\alpha]}$ and $K_{klU}^{[\alpha]}$ in (29) and (38), respectively, are also determined. The last step would be taking an $N$ by which $K_{ilU}^{[\alpha]}/N^{l+2}$ and $K_{klU}^{[\alpha]}/N^{l+2}$ are as small as we wish. Following this kind of guideline undoubtedly leads to the analysis of the $L_\infty$-induced norm of the continuous-time LTI system (5) with any degree of accuracy.

Furthermore, it would be worthwhile to note that the computation methods developed in this paper can be readily applied to general continous-time LTI systems regardless of invertible or non-invertible case of the systems since they do not involve an inverse matrix computation.

*Remark 5:* Our preceding studies [19], [20] relevant to the piecewise constant ($l = 0$) and piecewise linear ($l = 1$) approximation schemes under $\alpha = 0$ can be regarded as special cases of (42) and (43) except for the following issue: the particular situation for the input approximation approach with the piecewise constant approximation scheme allowed us to derive a slightly improved lower bound $|F_{HNi0}^{-[\alpha]}|_\infty$ instead of the corresponding leftmost side of (42).

## VI. NUMERICAL EXAMPLE

This section is devoted to examining the effectiveness of the developed methods for computing upper/lower bounds through comparison with the results of the well-known fixed stepsize Runge-Kutta 4th-order method (RK4) [23]. More precisely, we compute the impulse state response of (5) by solving the state equation with RK4, together with the integral of the absolute value of the impulse output response over the same interval $[0, H]$ as that underlying $\mathcal{F}_H^-$ in our arguments. In particular, the integral computation is recast as solving an associated differential equation (corresponding to that of an integrator), which is actually solved with RK4 simultaneously with the above state equation, where the stepsize is determined as $h := H/N$; this treatment leads to the accuracy of RK4 in the same order of $1/N^4$ as that in the arguments of the piecewise cubic approximation scheme in this paper and makes the comparison fair when our upper/lower bounds are computed with the same fast-lifting parameter $N$.

Note, however, that RK4 do not care about $\mathcal{F}_H^+$ in (10) and (13). This observation indicates that the upper bound (UB) and lower bound (LB) in Theorem 1 or Theorem 2 (about $\|\mathcal{F}_H^-\|_\infty$) rather than those in Theorem 3 (about $\|\mathcal{F}\|_\infty$) are the adequate values that are to be compared with the results of RK4; our interest arises from the fact that RK4 *cannot give fully theoretical upper and lower bounds* and is directed to observing if the results of RK4 lies within the upper and lower bounds for the methods in this paper.

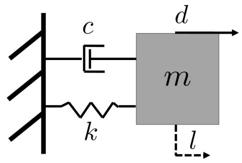


**FIGURE 2.** Mass-spring-damper system.

**TABLE 3.** Computation results for $K_{il}^{[\alpha]}$ ($l = 2, 3$).

| $\alpha$ | 0.0 | 0.2 | 0.4 | 0.5 | 0.7 | 0.9 |
|---|---|---|---|---|---|---|
| $K_{i2}^{[\alpha]}$ | 12.9833 | 1.1225 | 0.1146 | 0.0557 | 0.3304 | 4.1554 |
| $K_{i3}^{[\alpha]}$ | 60.3798 | 2.1752 | 0.0682 | 0.0199 | 0.3261 | 12.2808 |

**TABLE 4.** Computation results for $K_{kl}^{[\alpha]}$ ($l = 2, 3$).

| $\alpha$ | 0.0 | 0.2 | 0.4 | 0.5 | 0.7 | 0.9 |
|---|---|---|---|---|---|---|
| $K_{k2}^{[\alpha]}$ | 0.0417 | 0.0171 | 0.0065 | 0.0052 | 0.0103 | 0.0273 |
| $K_{k3}^{[\alpha]}$ | 0.0083 | 0.0027 | 0.0007 | 0.0005 | 0.0014 | 0.0049 |

Let us consider the mass-spring-damper system shown in Fig. 2, where $m$, $k$, $c$, $l$ and $d$ denote the mass, spring constant, damper constant, displacement of the mass from its equilibrium point and unknown disturbance in $L_\infty$ (i.e., bounded persistent disturbance) affecting the mass $m$, respectively. This system has been regarded as one of the most useful models for describing real systems since a number of mechanical systems can be represented by this system, and its motion is given by the second-order transfer function

$$L(s) = \frac{1}{ms^2 + cs + k} D(s) \quad (44)$$

where $L(s)$ and $D(s)$ denote the Laplace transforms of $l$ and $d$, respectively. We consider the case when $m = k = 1$ and $c = 0.6$ and thus

$$A = \begin{bmatrix} -0.6 & -1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad D = 0 \quad (45)$$

**TABLE 5.** Computation results with $\alpha = 0.5$ and comparison to RK4 (analytical value $\nu = 2.1863\,6798$, $K_{Hq} = 2.8 \times 10^{-8}$).

| $N$ | 400 | 2000 |
|---|---|---|
| i) Input Approximation ($l = 2$) | | |
| UB in Theorem 3 | *2.1891 7714* | *2.1863 8513* |
| UB in Theorem 1 | 2.1891 7711 | 2.1863 8510 |
| $\Vert F_{HNi2}^{-[0.5]}\Vert_\infty$ | 2.1863 6768 | 2.1863 6798 |
| LB in both Theorems 1 and 3 | *2.1835 5825* | *2.1863 5085* |
| ii) Input Approximation ($l = 3$) | | |
| UB in Theorem 3 | *2.1865 6848* | *2.1863 6825* |
| UB in Theorem 1 | 2.1865 6845 | 2.1863 6822 |
| $\Vert F_{HNi3}^{-[0.5]}\Vert_\infty$ | 2.1863 6797 | 2.1863 6798 |
| LB in both Theorems 1 and 3 | *2.1861 6749* | *2.1863 6773* |
| iii) Kernel Approximation ($l = 2$) | | |
| UB in Theorem 3 | *2.1866 3215* | *2.1863 6961* |
| UB in Theorem 2 | 2.1866 3212 | 2.1863 6958 |
| $\Vert F_{HNk3}^{-[0.5]}\Vert_\infty$ | 2.1863 6955 | 2.1863 6798 |
| LB in both Theorems 2 and 3 | *2.1861 0699* | *2.1863 6638* |
| iv) Kernel Approximation ($l = 3$) | | |
| UB in Theorem 3 | *2.1863 7209* | *2.1863 6801* |
| UB in Theorem 2 | 2.1863 7206 | 2.1863 6798 |
| $\Vert F_{HNk3}^{-[0.5]}\Vert_\infty$ | 2.1863 6681 | 2.1863 6798 |
| LB in both Theorems 2 and 3 | *2.1863 6156* | *2.1863 6797* |
| v) result with RK4 | 2.1855 8887 | 2.1863 6836 |

for which the $L_1$ norm of $\nu$ of its impulse response is given by the analytic expression $(1 + \exp(-\zeta\pi/\sqrt{1-\zeta^2}))/(1 - \exp(-\zeta\pi/\sqrt{1-\zeta^2}))$ with $\zeta = 0.3$, and this leads to $\nu = 2.1863\,6797$.

We first take $q = 2$ together with $H = 80$, by which we obtain $|\exp(Aq)|_\infty = 0.8873$ and $K_{Hq} = 2.8 \times 10^{-8}$. Regarding an adequate choice of $\alpha$ over $[0, 1]$ relevant to $\mathbf{B}_{il}^{[\alpha]}$ and $\mathbf{B}_{kl}^{[\alpha]}$, the computation results for $K_{il}^{[\alpha]}$ and $K_{kl}^{[\alpha]}$ in (28) and (37), respectively, are given in Tables 3 and 4, respectively, for reference. From Tables 3 and 4, it can be confirmed that $K_{kl}^{[\alpha]}$ is quite smaller than $K_{il}^{[\alpha]}$ under the same parameters $l$ and $\alpha$. Furthermore, we can see from these tables that both $K_{il}^{[\alpha]}$ and $K_{kl}^{[\alpha]}$ achieve their minimum values at $\alpha = 0.5$ and thus we take $\alpha = 0.5$ when we compute $\Vert\mathcal{F}_H^-\Vert_\infty$. Then, we consider the case with $N = 400$ and $N = 2000$. The associated computation results are given in Table 5, where those values relevant to Theorem 3 with $\Vert\mathcal{F}_H^+\Vert_\infty$ taken into account are shown in italic.

It could be observed from Table 5 that all the computation results for $\Vert F_{HNi2}^{-[0.5]}\Vert_\infty$, $\Vert F_{HNi3}^{-[0.5]}\Vert_\infty$, $\Vert F_{HNk2}^{-[0.5]}\Vert_\infty$ and $\Vert F_{HNk3}^{-[0.5]}\Vert_\infty$ are closer to the exact value $\nu$ than those for RK4 at the same parameter $N$. We next note the case of $N = 400$ and examine whether the result for RK4 lies within the ranges of the upper and lower bounds obtained by the arguments in this paper (i.e., Theorems 1 and 2). We then see that even though it lies within the range for the input approximation approach with the piecewise quadratic approximation scheme, it fails to do so for other three methods with improved tighter bounds. Even though the unfavorable situation for the result of RK4 slightly improves in the case of $N = 2000$, it still fails to lie within the ranges for the piecewise cubic approximation scheme (which possesses the same convergence rate with RK4) both under the input approximation and kernel approximation approaches. Thus, we could see from these observations that the methods developed in this paper, particularly those with the piecewise cubic approximation scheme, are superior to RK4 for the computation of the $L_\infty$-induced norm of continuous-time LTI systems.

In addition, the developed methods are quite important because, unlike RK4, they are aimed at providing upper and lower bounds on the $L_\infty$-induced norm itself, as shown in Theorem 3. In this context, we can further confirm from the values in italic in Table 5 that the exact value $\nu$ for the $L_\infty$-induced norm always lies within the range for each of the four methods developed in this paper. These observations undoubtedly suggest that our methods can be very promising alternatives to a naive method that directly deals with the impulse response numerically.

## VII. CONCLUDING REMARKS

Regarding performance analysis for bounded persistent disturbances of continuous-time linear time-invariant (LTI) systems, this paper tackled the computation of upper and lower bounds on the $L_\infty$-induced norm of continuous-time LTI systems. Specifically, under the input approximation and kernel approximation approaches, we developed piecewise quadratic approximation and piecewise cubic approximation schemes for achieving better convergence rates than the existing methods with the piecewise constant approximation and piecewise linear approximation schemes. More precisely, it was shown that the error bounds in the treatment of $\|\mathcal{F}_H^-\|_\infty$ under the fast-lifting parameter $N$ converge to 0 at the rates of $1/N^3$ and $1/N^4$ in the piecewise quadratic approximation scheme and the piecewise cubic approximation scheme, respectively, under both the input approximation and kernel approximation approaches. However, the contributions of this paper are not limited simply to the use of higher-order approximation. More specifically, this paper established generalized arguments for enhanced use of the Taylor expansion of relevant functions in our approximation approaches, which can lead to improvement without increasing the approximation order; our preceding studies with piecewise constant approximation and piecewise linear approximation schemes [19], [20] only used the Taylor expansion at the beginning of an interval. Furthermore, we demonstrated through a numerical example that the methods developed in this paper, particularly those with the piecewise cubic approximation scheme, are superior to a naive method such as the Runge-Kutta 4th-order method that directly deals with the impulse response numerically. Furthermore, because the $L_\infty$-induced norm is independent of time-delay contained in input and/or output of a system, the computation methods proposed in this paper could be expected to be widely used in a number of practical systems.

It would be also worthwhile to remark that constructing the $l$th-order approximants $\mathbf{B}_{il}$ and $\mathbf{B}_{kl}$ to $\mathbf{B}$ with $l \geq 4$ can be studied by following the same line of the arguments. However, the theoretical interest in taking such $l$, particularly $l > 4$, is not clear since we are led to the (exact) computation of the $L_1[0,h)$ norms of $l$th-order polynomials; this is not theoretically appealing because of its relevance to the finding of real roots of higher-order polynomials, for which no analytical method exists when $l > 4$. Hence, the only remaining issue of theoretical interest from the the present authors' viewpoint is studying whether the piecewise quartic approximation approach could be constructed in a similar way. Since it seems highly likely that we require rather involved manipulations, a rigorous study remains open.

## APPENDIX

This appendix is devoted to providing the rationale for the specific choice of the functions $f_{lp}^{[\alpha]}(\tau)$ $(l = 2, 3;\ p = 0, \ldots, l;\ \alpha \in [0, 1])$ given in (21) and (22) together with the proofs of Lemmas 1–4.

## APPENDIX A
## THE RATIONALE FOR SPECIFIC CHOICE OF FUNCTIONS

We aim at showing that the scalar-valued functions $f_{lp}^{[\alpha]}(\tau)$ $(l = 2, 3;\ p = 0, \ldots, l;\ \alpha \in [0, 1])$ in (21) and (22) are such that

$$\|\mathcal{F}_{HN}^- - \mathcal{F}_{HNil}^{-[\alpha]}\|_\infty = \sup_{\|u\|_\infty \leq 1} \left\| \sum_{m=1}^{N} C(A_d)^{m-1}(\mathbf{B} - \mathbf{B}_{il}^{[\alpha]})u^{(m)} \right\|_\infty \quad (46)$$

converges to 0 at the rates of $1/N^{l+1}$ $(l = 2, 3)$ as $N$ tends to infinity. To this end, it suffices to take such functions $f_{lp}^{[\alpha]}(\tau)$ $(l = 2, 3;\ p = 0, \ldots, l)$ making

$$\|C(A_d)^{m-1}(\mathbf{B} - \mathbf{B}_{il}^{[\alpha]})\|_\infty \quad (47)$$

converges to 0 in the orders of $1/N^{l+2}$ $(l = 2, 3)$ for each $m$. Since $|C(A_d)^{m-1}|_\infty$ has an upper bound independent of $N$ and $m$, it suffices for us to find such functions for which $\|\mathbf{B} - \mathbf{B}_{il}^{[\alpha]}\|_\infty$ is bounded by the order of $1/N^{l+2}$, which is equivalent to the order of $h^{l+2}$. Here, note that

$$\begin{aligned}
&\|\mathbf{B} - \mathbf{B}_{il}^{[\alpha]}\|_\infty \\
&= \sup_{\|u\|_\infty \leq 1} \left|(\mathbf{B}u - \mathbf{B}\mathbf{J}_l^{[\alpha]}u)\right|_\infty \\
&= \sup_{\|u\|_\infty \leq 1} \left| \int_0^h \exp(A\theta)Bu(\theta)d\theta - \int_0^h \exp(A\theta)B \right. \\
&\quad \left. \times \left( \sum_{p=0}^{l} (\theta - \alpha h)^p \int_0^h f_{lp}^{[\alpha]}(\tau)u(\tau)d\tau \right) d\theta \right|_\infty \\
&= \sup_{\|u\|_\infty \leq 1} \left| \int_0^h \exp(A\theta)Bu(\theta)d\theta - \sum_{p=0}^{l} \left( \int_0^h \exp(A\theta) \right. \right. \\
&\quad \left. \left. \times B(\theta - \alpha h)^p d\theta \right) \int_0^h f_{lp}^{[\alpha]}(\tau)u(\tau)d\tau \right|_\infty \\
&= \sup_{\|u\|_\infty \leq 1} \left| \int_0^h \left( \exp(A\tau)B - \sum_{p=0}^{l} f_{lp}^{[\alpha]}(\tau)B_{pd}^{[\alpha]} \right) u(\tau)d\tau \right|_\infty
\end{aligned} \quad (48)$$

where $B_{pd}^{[\alpha]}$ is described as (20). This immediately leads to

$$\|\mathbf{B} - \mathbf{B}_{il}^{[\alpha]}\|_\infty \leq \int_0^h \left| \exp(A\tau)B - \sum_{p=0}^{l} f_{lp}^{[\alpha]}(\tau)B_{pd}^{[\alpha]} \right|_\infty d\tau \quad (49)$$

It readily follows from the Taylor expansions of $\exp(A\tau)$ and $\exp(A\theta)$ in the definition of $B_{pd}^{[\alpha]}$ in (20) at $\alpha h$ that

$$\begin{aligned}
&\exp(A\tau)B - \sum_{p=0}^{l} f_{lp}^{[\alpha]}(\tau)B_{pd}^{[\alpha]} \\
&= \sum_{p=0}^{\infty} \exp(A\alpha h)\frac{A^p(\tau - \alpha h)^p}{p!}B \\
&\quad - \sum_{p=0}^{l} f_{lp}^{[\alpha]}(\tau) \sum_{j=0}^{\infty} \exp(A\alpha h)\frac{A^j h^{p+j+1}}{(p+j+1)\cdot j!} \\
&\quad \times \{(1-\alpha)^{p+j+1} - (-\alpha)^{p+j+1}\}B
\end{aligned} \quad (50)$$

To satisfy the condition about the convergence order mentioned above, an easy way is to assume that $f_{lp}^{[\alpha]}(\tau)$ is given in the form

$$f_{lp}^{[\alpha]}(\tau) = \sum_{j=0}^{l} \frac{c_{lpj}^{[\alpha]}}{h^{p+j+1}}(\tau - \alpha h)^j \quad (51)$$

where $c_{lpj}^{[\alpha]}$ is independent of $h$ but is dependent on $\alpha$, and find appropriate $c_{lpj}^{[\alpha]}$ $(p, j = 0, \ldots, l)$ (almost) nullifying all the coefficients about $(\tau - \alpha h)^p$ $(p = 0, \ldots, l)$ of the function in (50). More precisely, the coefficients $c_{lpj}^{[\alpha]}$ $(p, j = 0, \ldots, l)$ such that the function

$$\sum_{j=0}^{l} \frac{A^j(\tau-\alpha h)^j}{j!} - \sum_{p=0}^{l} \Big\{ \Big( \sum_{j=0}^{l} \frac{c_{lpj}^{[\alpha]}}{h^{p+j+1}} (\tau-\alpha h)^j \Big) \times \Big( \sum_{i=0}^{l} \frac{A^i h^{p+i+1}}{(p+i+1)\cdot i!} \cdot \{(1-\alpha)^{p+i+1} - (-\alpha)^{p+i+1}\} \Big) \Big\} \tag{52}$$

(obtained essentially by truncating the two infinite sums in (50)) is identically zero make the integral in (49) obviously bounded by the order of $h^{l+2}$. Such coefficients of the functions $f_{lp}^{[\alpha]}(\tau)$ $(l = 2, 3;\ p = 0, \ldots, l)$ in (21) and (22) can be determined as follows. For a fixed $j \in \{0, \ldots, l\}$, let us consider the $(l+1)$ linear equations with $i \in \{0, \ldots, l\}$ such that

$$\sum_{p=0}^{l} \frac{c_{lpj}^{[\alpha]}}{h^{p+j+1}} \cdot \frac{h^{p+i+1}\cdot\{(1-\alpha)^{p+i+1} - (-\alpha)^{p+i+1}\}}{(p+i+1)\cdot i!} = \begin{cases} \frac{1}{j!} & (\text{if } i = j) \\ 0 & (\text{if } i \neq j) \end{cases} \tag{53}$$

By solving the above linear equations in $c_{lpj}$ $(l, p = 0, \ldots, l)$ for all $j \in \{0, \ldots, l\}$ under the given $l$, we obtain those in Tables 1 and 2 for $l = 2$ and $l = 3$, respectively.

## APPENDIX B
## PROOFS OF LEMMAS

Because the proofs of those in Section IV proceed in essentially the same way relevant to those in Section III, only the proofs of Lemmas 1 and 2 are given in this appendix.

### 1) PROOF OF LEMMA 1

By using (19), we have

$$\mathcal{F}_{HNil}^{-[\alpha]}\check{u} = Du^{(1)}(0) + \sum_{m=1}^{N} C(A_d)^{m-1}\mathbf{B}_{il}^{[\alpha]}u^{(m)} \tag{54}$$

By the definition of $\mathbf{B}_{il}^{[\alpha]}$, we can see that

$$\begin{aligned} &C(A_d)^{m-1}\mathbf{B}_{il}^{[\alpha]}u^{(m)} \\ &= C(A_d)^{m-1}\mathbf{B}\mathbf{J}_{il}^{[\alpha]}u^{(m)} \\ &= C(A_d)^{m-1}\int_0^h \exp(A\theta)B\sum_{p=0}^{l}(\theta-\alpha h)^p \\ &\quad \times \Big(\int_0^h \sum_{j=0}^{l}\frac{c_{lpj}^{[\alpha]}}{h^{p+j+1}}(\tau-\alpha h)^j u(\tau)d\tau\Big)d\theta \\ &= C(A_d)^{m-1}\int_0^h \sum_{p=0}^{l}\Big(B_{pd}^{[\alpha]}\sum_{j=0}^{l}\frac{c_{lpj}^{[\alpha]}}{h^{p+j+1}} \\ &\quad \times (\tau-\alpha h)^j\Big)u(\tau)d\tau \\ &= \int_0^h C(A_d)^{m-1}\sum_{j=0}^{l}G_{lj}^{[\alpha]}(\tau-\alpha h)^j u(\tau)d\tau \end{aligned} \tag{55}$$

Note that the integrand involves the function used in defining $T_{lm}^{[\alpha]}$ $(l = 2, 3;\ m = 1, \ldots, N)$. Hence, by the property of $L_\infty[0, h)$ and the definition of $F_{HNil}^{-[\alpha]}$, it follows that $\|\mathcal{F}_{HNil}^{-[\alpha]}\|_\infty$ coincides with the $\infty$-norm of the finite-dimensional matrix $F_{HNil}^{-[\alpha]}$ given in Lemma 1. This completes the proof of Lemma 1.

### 2) PROOF OF LEMMA 2

We first note that

$$\begin{aligned} \|\mathcal{F}_{HN}^{-} - \mathcal{F}_{HNil}^{-[\alpha]}\|_\infty &= \sup_{\|\check{u}\|_\infty \le 1} \Big|(\mathcal{F}_{HN}^{-} - \mathcal{F}_{HNil}^{-[\alpha]})\check{u}\Big|_\infty \\ &= \sup_{\|\check{u}\|_\infty \le 1} \Big|C_{dN}\overline{(\mathbf{B}-\mathbf{B}_{il}^{[\alpha]})\check{u}}\Big|_\infty \\ &\le |C_{dN}|_\infty \cdot \sup_{\|u\|_\infty \le 1}\Big|(\mathbf{B}-\mathbf{B}_{il}^{[\alpha]})u\Big|_\infty \\ &= |C_{dN}|_\infty \cdot \Big\|\mathbf{B}-\mathbf{B}_{il}^{[\alpha]}\Big\|_\infty \end{aligned} \tag{56}$$

Here, note that $\|\mathbf{B}-\mathbf{B}_{il}^{[\alpha]}\|_\infty$ is evaluated in (49), in the RHS of which the functions $f_{lp}^{[\alpha]}(\tau)$ $(l = 2, 3;\ p = 0, \ldots, l)$ (in particular, the coefficients $c_{lpj}^{[\alpha]}$ $(l = 2, 3;\ p = 0, \ldots, l)$) have been determined to make the function in (52) identically zero. If we recall that (52) was obtained essentially by truncating the two infinite sums in (50), it readily follows that

$$\begin{aligned} &\|\mathbf{B}-\mathbf{B}_{il}^{[\alpha]}\|_\infty \\ &\le \int_0^h \Big|\exp(A\tau)B - \sum_{p=0}^{l} f_{lp}^{[\alpha]}(\tau)B_{pd}^{[\alpha]}\Big|_\infty d\tau \\ &\le \int_0^h \Big|\sum_{p=l+1}^{\infty}\exp(A\alpha h)\frac{A^p(\tau-\alpha h)^p}{p!}B\Big|_\infty d\tau \\ &\quad + \int_0^h \Big|\sum_{p=0}^{l} f_{lp}^{[\alpha]}(\tau)\sum_{j=l+1}^{\infty}\exp(A\alpha h)\frac{A^j h^{p+j+1}}{(p+j+1)\cdot j!} \\ &\quad \times \{(1-\alpha)^{p+j+1} - (-\alpha)^{p+j+1}\}B\Big|_\infty d\tau \\ &\le |\exp(A\alpha h)A^{l+1}B|_\infty \int_0^h \sum_{j=0}^{\infty}\frac{|A|_\infty^j \cdot |\tau-\alpha h|^{l+j+1}}{(l+j+1)!}d\tau \\ &\quad + |\exp(A\alpha h)A^{l+1}B|_\infty \times \int_0^h \sum_{p=0}^{l}|f_{lp}^{[\alpha]}(\tau)| \\ &\quad \times \sum_{j=0}^{\infty}\frac{|A|_\infty^j h^{l+p+j+2}\cdot\{(1-\alpha)^{l+p+j+2}+\alpha^{l+p+j+2}\}}{(l+p+j+2)\cdot(l+j+1)!}d\tau \end{aligned} \tag{57}$$

Here, we can show that

$$\int_0^h \sum_{j=0}^{\infty}\frac{|A|_\infty^j \cdot |\tau-\alpha h|^{l+j+1}}{(l+j+1)!}d\tau \le \frac{((1-\alpha)^{l+2}+\alpha^{l+2})h^{l+2}}{(l+2)!}e^{|A|_\infty h} \tag{58}$$

$$\sum_{j=0}^{\infty}\frac{|A|_\infty^j h^{l+p+j+2}\cdot\{(1-\alpha)^{l+p+j+2}+\alpha^{l+p+j+2}\}}{(l+p+j+2)\cdot(l+j+1)!} \le \frac{((1-\alpha)^{l+p+2}+\alpha^{l+p+2})h^{l+p+2}}{(l+p+2)\cdot(l+1)!}e^{|A|_\infty h} \tag{59}$$

where we used

$$\begin{aligned}
&\int_0^h |\tau - \alpha h|^{l+j+1} d\tau \\
&= \int_0^{\alpha h} (-\tau + \alpha h)^{l+j+1} d\tau + \int_{\alpha h}^{h} (\tau - \alpha h)^{l+j+1} d\tau \\
&= \frac{(\alpha h)^{l+j+2}}{l+j+2} + \frac{((1-\alpha)h)^{l+j+2}}{l+j+2} \qquad (60)
\end{aligned}$$

in (58). From these inequality and the definition of $f_{lp}^{[\alpha]}(\tau)$ given in (51), the left hand side of (57) is further bounded by

$$\begin{aligned}
&|\exp(A\alpha h)A^{l+1}B|_\infty \Bigg( \frac{((1-\alpha)^{l+2} + \alpha^{l+2}))h^{l+2}}{(l+2)!} e^{|A|_\infty h} \\
&+ \sum_{p=0}^{l} \sum_{j=0}^{l} \Bigg( \frac{|c_{lpj}^{[\alpha]}|}{h^{p+j+1}} \frac{((1-\alpha)^{j+1} + \alpha^{j+1})h^{j+1}}{j+1} \\
&\times \frac{((1-\alpha)^{l+p+2} + \alpha^{l+p+2})h^{l+p+2}}{(l+p+2)\cdot(l+1)!} e^{|A|_\infty h} \Bigg) \Bigg) \\
&= |\exp(A\alpha h)A^{l+1}B|_\infty \cdot e^{|A|_\infty h} \cdot h^{l+2} \cdot K_{il}^{[\alpha]} \qquad (61)
\end{aligned}$$

This together with (56) proves the first assertion of Lemma 2 since $h = H/N$. The second assertion of Lemma 2 follows if we note

$$\begin{aligned}
&\frac{|C_{dN}|_\infty}{N} \cdot |\exp(A\alpha h)A^{l+1}B|_\infty \cdot e^{|A|_\infty h} \\
&\leq |C|_\infty \cdot e^{|A|_\infty (N-1)h} \cdot |A^{l+1}B|_\infty \cdot e^{|A|_\infty \alpha h} \cdot e^{|A|_\infty h} \\
&= |C|_\infty \cdot |A^{l+1}B|_\infty \cdot e^{|A|_\infty (H+\alpha h)} \\
&\leq |C|_\infty \cdot |A^{l+1}B|_\infty \cdot e^{|A|_\infty (1+\alpha)H} \qquad (62)
\end{aligned}$$

## REFERENCES


[1] K. Zhou, J. C. Doyle, and K. Glover, *Robust and Optimal Control*. London, U.K.: Pearson, 1995.

[2] K. Zhou and J. C. Doyle, *Essentials of Robust and Optimal Control*. London, U.K.: Pearson, 1997.

[3] J. C. Doyle and G. Stein, "Multivariable feedback design: Concepts for a classical/modern synthesis," *IEEE Trans. Autom. Control*, vol. AC-26, no. 1, pp. 4–16, Feb. 1981.

[4] J. C. Doyle, K. Glover, P. P. Khargonekar, and B. A. Francis, "State-space solutions to standard $H_2$ and $H_\infty$ control problems," *IEEE Trans. Autom. Control*, vol. 34, no. 8, pp. 831–847, Aug. 1989.

[5] B.-C. Chang and J. Pearson, "Optimal disturbance reduction in linear multivariable systems," *IEEE Trans. Autom. Control*, vol. AC-29, no. 10, pp. 880–887, Oct. 1984.

[6] B. A. Francis, J. W. Helton, and G. Zames, "$H^\infty$-optimal feedback controllers for linear multivariable systems," *IEEE Trans. Autom. Control*, vol. AC-29, no. 10, pp. 888–900, Oct. 1984.

[7] S. Boyd, V. Balakrishnan, and P. Kabamba, "A bisection method for computing the $H_\infty$ norm of a transfer matrix and related problems," *Math. Control, Signals, Syst.*, vol. 2, no. 3, pp. 207–219, Sep. 1989.

[8] J. H. Kim, S.-M. Hur, and Y. Oh, "Performance analysis for bounded persistent disturbances in PD/PID-controlled robotic systems with its experimental demonstrations," *Int. J. Control*, vol. 91, no. 3, pp. 688–705, Mar. 2018.

[9] J. H. Kim, J. Lee, and Y. Oh, "A theoretical framework for stability regions for standing balance of humanoids based on their LIPM treatment," *IEEE Trans. Syst., Man, Cybern. Syst.*, early access, Jul. 31, 2020, doi: 10.1109/TSMC.2018.2855190.

[10] Z. Chen and J. Huang, "Attitude tracking and disturbance rejection of rigid spacecraft by adaptive control," *IEEE Trans. Autom. Control*, vol. 54, no. 3, pp. 600–605, Mar. 2009.

[11] N. Prabhakar, A. Painter, R. Prazenica, and M. Balas, "Trajectory-driven adaptive control of autonomous unmanned aerial vehicles with disturbance accommodation," *J. Guid., Control, Dyn.*, vol. 41, no. 9, pp. 1976–1989, Sep. 2018.

[12] P. Trodden and A. Richards, "Distributed model predictive control of linear systems with persistent disturbances," *Int. J. Control*, vol. 83, no. 8, pp. 1653–1663, Aug. 2010.

[13] J. Wen and F. Liu, "Receding horizon control for constrained Markovian jump linear systems with bounded disturbance," *J. Dyn. Syst., Meas., Control*, vol. 133, no. 1, pp. 011005-1–011005-10, Jan. 2011.

[14] E. Garcia and P. J. Antsaklis, "Output feedback networked control with persistent disturbance attenuation," *Syst. Control Lett.*, vol. 62, no. 10, pp. 943–948, Oct. 2013.

[15] M. Vidyasagar, "Optimal rejection of persistent bounded disturbances," *IEEE Trans. Autom. Control*, vol. AC-31, no. 6, pp. 527–534, Jun. 1986.

[16] M. A. Dahleh and J. B. Pearson, "$L^1$-optimal compensators for continuous-time systems," *IEEE Trans. Autom. Control*, vol. AC-32, no. 10, pp. 889–895, Oct. 1987.

[17] Y. Ohta, H. Maeda, and S. Kodama, "Rational approximation of $L_1$ optimal controllers for SISO systems," *IEEE Trans. Autom. Control*, vol. 37, no. 11, pp. 1683–1691, Nov. 1992.

[18] N. K. Rutland and P. G. Lane, "Computing the 1-norm of the impulse response of linear time-invariant systems," *Syst. Control Lett.*, vol. 26, no. 3, pp. 211–221, Oct. 1995.

[19] J. H. Kim and T. Hagiwara, "Computing the $L_\infty[0, h)$-induced norm of a compression operator via fast-lifting," *Syst. Control Lett.*, vol. 67, no. 1, pp. 1–8, May 2014.

[20] J. H. Kim and T. Hagiwara, "Computing the $L_\infty$-induced norm of linear time-invariant systems via kernel approximation and its comparison with input approximation," *IET Control Theory Appl.*, vol. 9, no. 5, pp. 700–709, Mar. 2015.

[21] T. Hagiwara and H. Umeda, "Modified fast-sample/fast-hold approximation for sampled-data system analysis," *Eur. J. Control*, vol. 14, no. 4, pp. 286–296, 2008.

[22] Y. W. Choi, J. H. Kim, and T. Hagiwara, "L1 analysis of LTI systems via piecewise higher-order approximation," in *Proc. IEEE Multi-Conf. Syst. Control*, Sep. 2015, pp. 1410–1415.

[23] W. H. Press, B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling, *Numerical Recipes in C: The Art of Scientific Computing*. Cambridge, U.K.: Cambridge Univ. Press, 1988.


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