GLOBAL SOLUTIONS OF NONLINEAR TRANSPORT EQUATIONS FOR CHEMOSENSITIVE MOVEMENT*

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Abstract. A widespread phenomenon in moving microorganisms and cells is their ability to reorient themselves depending on changes of concentrations of certain chemical signals. In this paper we discuss kinetic models for chemosensitive movement, which also takes into account evaluations of gradient fields of chemical stimuli which subsequently influence the motion of the respective microbiological species. The basic type of model was discussed by Alt [*J. Math. Biol.*, 9 (1980), pp. 147–177], [*J. Reine Angew. Math.*, 322 (1981), pp. 15–41] and by Othmer, Dunbar, and Alt [*J. Math. Biol.*, 26 (1988), pp. 263–298]. Chalub et al. rigorously proved that, in three dimensions, these kinds of kinetic models lead to the classical Keller–Segel model as its drift-diffusion limit when the equation for the chemo-attractant is of elliptic type [*Monatsh. Math.*, 142 (2004), pp. 123–141], [*On the Derivation of Drift-Diffusion Model for Chemotaxis from Kinetic Equations*, ANUM preprint 14/02, Vienna Technical University, 2002]. In [H. Hwang, K. Kang, and A. Stevens, *Drift-diffusion limits of kinetic models for chemotaxis: A generalization*, Discrete Contin. Dyn. Syst. Ser. B., to appear] it was proved that the macroscopic diffusion limit exists in both two and three dimensions also when the equation of the chemo-attractant is of parabolic type. So far in the rigorous derivations, only the density of the chemo-attractant was supposed to influence the motion of the chemosensitive species. Here we show that in the macroscopic limit some types of evaluations of gradient fields of the chemical stimulus result in a change of the classical parabolic Keller–Segel model for chemotaxis. Under suitable structure conditions, global solutions for the kinetic models can be shown.

Key words. chemosensitive movement, sensing of gradient fields, nonlinear transport equations, global solutions, drift-diffusion limit, Keller–Segel model

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1. Introduction. The starting point of our considerations is the classical chemotaxis model as discussed by Keller and Segel (see [14] and [15]). This system is of advection-diffusion type and consists of two coupled parabolic equations:

$$(1.1) \qquad \frac{\partial \rho}{\partial t} = \nabla \cdot (D(\rho, S)\nabla \rho - \chi(\rho, S)\rho \nabla S),$$

$$(1.2) \qquad \tau \frac{\partial S}{\partial t} = D_0 \Delta S + \alpha \rho - \beta S, \quad \alpha, \beta, \tau \geq 0.$$

Here $\rho = \rho(x, t)$ denotes the density of chemotactic cells and $S = S(x, t)$ is the density of the chemo-attractant. The cells are attracted by the chemical, and χ denotes their chemotactic sensitivity. The first rigorous derivation of the macroscopic chemotaxis equations from microscopic models, namely, interacting stochastic many particle systems, was given in [21]. In [11] a survey about known results on existence of global solutions and finite time blowup for this type of model was given.

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In [3] a kinetic model for (1.1) was discussed coupled with the Poisson equation without decay term

$$-\Delta S = \alpha \rho. \tag{1.3}$$

In [3, p. 3] the following kinetic equation for the oriented cell density $f = f(x, v, t) \geq 0$ was considered:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \int_V (T[S]f' - T^*[S]f)dv', \tag{1.4}$$

where x, v, and t indicate position, velocity, and time, respectively. Here the abbreviations $f' = f(x, v', t)$, $T[S] = T[S](x, v, v', t)$, and $T^*[S] = T[S](x, v', v, t)$ are used. The first term on the right-hand side of (1.4) describes the turning into direction v, and the second term the turning away from v. The cell density ρ fulfills

$$\rho(x, t) = \int_V f(x, v, t)dv,$$

where V is the set of admissible velocities which is assumed to be compact.

Using stochastic models for the motion of bacteria and leukocytes, Alt derived (1.1) from a transport equation similar to (1.4) [1, section 8], [2, section 3]. Later a general formulation of this velocity-jump process was presented and studied in [18, section 3]. In [10] and [19] Othmer and Hillen studied the formal diffusion limit of a transport equation of (1.4) by moment expansions, which generalizes parts of Alt's earlier works [1], [2]. A hyperbolic scaling and its formal limit were discussed in [6].

Based on [19] a rigorous proof of the macroscopic limit was given in [3]. After using diffusive scaling of time and space, the nondimensional form of (1.4) leads to [3, p. 4]

$$\epsilon^2 \frac{\partial f_\epsilon}{\partial t} + \epsilon v \cdot \nabla_x f_\epsilon = -\mathcal{T}_\epsilon[S_\epsilon](f_\epsilon), \quad x \in \mathbb{R}^n, \ v \in V, \ t > 0, \tag{1.5}$$

where

$$\mathcal{T}_\epsilon[Z](g) = \int_V (T_\epsilon^*[Z]g - T_\epsilon[Z]g')dv'.$$

The diffusion limit $\epsilon \to 0$ was studied for initial conditions

$$f_\epsilon(x, v, 0) \equiv f_0(x, v), \quad x \in \mathbb{R}^n, \ v \in V, \tag{1.6}$$

with (1.5) coupled to (1.3) for the chemo-attractant. In [3] it was shown that the coupled nonlinear system (1.5), (1.6), and (1.3) resulted in Keller–Segel-type equations for chemotaxis as its macroscopic drift-diffusion limit under suitable conditions on the turning kernel in three dimensions (compare, e.g., [3, Theorem 5] and [4, Theorem 2]). In [3] and [4] also global solutions were proved for suitable turning kernels for fixed $\epsilon > 0$.

In [12], as an extension of [3], the authors proved that such kinetic models have a macroscopic diffusion limit in both two and three dimensions also when the equation of the chemo-attractant is of parabolic type, i.e., $\tau > 0$, which is the original version of the chemotaxis model. An independent related result was given in [5].

In this article, we consider turning kernels depending not only on S but also on ∇S, as formally discussed, among others, in [22] and [19], i.e.,

$$(1.7) \qquad \epsilon^2 \frac{\partial f_\epsilon}{\partial t} + \epsilon v \cdot \nabla_x f_\epsilon = -\mathcal{T}_\epsilon[S_\epsilon, \nabla S_\epsilon](f_\epsilon), \quad x \in \mathbb{R}^n, \ v \in V, \ t > 0,$$

with initial condition (1.6) coupled to

$$(1.8) \qquad \tau \frac{\partial S_\epsilon}{\partial t} = \Delta S_\epsilon + \alpha \rho_\epsilon - \beta S_\epsilon, \quad \tau \geq 0, \ \alpha > 0, \ \beta \geq 0,$$

where

$$(1.9) \qquad \rho_\epsilon = \int_V f_\epsilon dv.$$

In what follows, for notational convenience, we write $\mathcal{T}_\epsilon[S_\epsilon]$ instead of $\mathcal{T}_\epsilon[S_\epsilon, \nabla S_\epsilon]$, unless any confusion is to be expected. Here we emphasize that the conditions on the turning kernel include also detection of spatial gradients of the chemo-attractant by the chemotactic cells. This behavior results under certain conditions in a macroscopic model which varies from the classical Keller–Segel system by additional higher order terms.

Our main result is that for suitable turning kernels which take into account the effects of gradient measurements of the chemical, global solutions exist also in two dimensions, and thus blowup of the solutions does not happen in finite time (compare Theorems 3.6 and 3.12 for the elliptic and parabolic cases, respectively).

The result is extended to three dimensions under some restrictions on the turning kernels. We also show the existence of a macroscopic diffusion limit of the kinetic model in two and three dimensions. More precisely, under similar assumptions on the turning kernel $T[S]$ as given in [3], we prove that the coupled nonlinear system (1.6), (1.7), and (1.8) converges to Keller–Segel-type equations and their variants for $\epsilon \to 0$ (compare Theorem 4.4). Our main tool is the potential estimate for S. In particular, in case the chemo-attractant equation is of elliptic type, i.e., $\tau = 0$ and in two dimensions, log-type estimates for the chemical S are used to obtain global existence for the kinetic model (similar techniques were used in [13, Lemma 4]).

The plan of this paper is as follows: In section 2, we introduce some notation used and briefly review the derivation of the macroscopic equation as presented in [3] and [12]. In section 3, we prove that the kinetic model (1.7)–(1.9) has a global solution for "suitable" turning kernels. In section 4, we prove the existence of the diffusion limit for a short time interval. In section 5 we give concrete examples on how the specific dependencies of the turning kernel result in different types of macroscopic equations.

2. Preliminaries. We first introduce some notation which will be used throughout this article and recall some of the observations presented in [3].

• By G we denote the Bessel potential, which is the fundamental solution of the differential operator $1 - \Delta$ in $\mathbb{R}^n$ (see [20, pp. 130–132]):

$$(2.1) \qquad G(x) = \frac{1}{4\pi} \int_0^\infty e^{-\pi \frac{|x|^2}{4s} - \frac{s}{4\pi}} s^{\frac{-n+2}{2}} \frac{ds}{s}.$$

• By Γ we denote the fundamental solution of the differential operator $\partial_t - \Delta_x + \beta$ in $\mathbb{R}^n \times \mathbb{R}_+$:

$$(2.2) \qquad \Gamma(x, t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp\left(-\frac{|x|^2}{4t} - \beta t\right).$$

• By $C = C(\alpha, \beta, \dots)$ we denote a constant depending on the prescribed quantities $\alpha, \beta, \dots$. The domain Ω considered in this article is $\mathbb{R}^n$, $n = 2, 3$.

To make this note self-contained, we review the formal derivation of the macroscopic equation from the kinetic model presented in [3] (compare the details in [3, pp. 5–7]). For simplicity we assume for a moment that $\tau = 1$, $\alpha = 1$, and $\beta = 1$ (other cases can be formally derived in a similar way without any difficulty). Since the integral of $\mathcal{T}_\epsilon[S](f)$ with respect to the velocity vanishes, we obtain the macroscopic conservation equation

$$(2.3) \qquad \frac{\partial \rho_\epsilon}{\partial t} + \nabla \cdot J_\epsilon = 0,$$

where $J_\epsilon(x, t) = \epsilon^{-1} \int_V v f_\epsilon(x, v, t) dv$ is the flux density. The turning kernel is assumed to have the following asymptotic expansion: $T_\epsilon[S] = T_0[S] + \epsilon T_1[S] + O(\epsilon^2)$. Then the turning operator can be expanded in a similar way and

$$\mathcal{T}_\epsilon[S](f) = \int_V (T_\epsilon^*[S]f - T_\epsilon[S]f')dv'.$$

By asymptotic expansion of $f_\epsilon = f_0 + \epsilon f_1 + O(\epsilon^2)$ and $S_\epsilon = S_0 + \epsilon S_1 + O(\epsilon^2)$, the equation for the leading order terms can be obtained from (1.7):

$$(2.4) \qquad \mathcal{T}_0[S_0](f_0) = 0, \quad S_0 = \rho_0 * \Gamma, \quad \rho_0 = \int_V f_0 dv.$$

Comparing coefficients in (1.7) results in

$$v \cdot \nabla_x f_0 = -\mathcal{T}_0[S_0](f_1) - \mathcal{T}_1[S_0](f_0) - \mathcal{T}_{0S}[S_0, S_1](f_0),$$

where $\mathcal{T}_{0S}[S_0, S_1]$ is part of the turning operator $\mathcal{T}$ and its kernel is the Fréchet derivative of T_0 with respect to S, evaluated at S_0 in the direction S_1. Here, we recall the assumptions on the leading order terms of the turning operator and two useful lemmas presented in [3, (A0), Lemma 1, Lemma 2, pp. 6–7].

Assumption 2.1. There exists a bounded velocity distribution $F(v) > 0$, such that $T_0^*[S]F = T_0[S]F'$ and

$$\int_V v F(v) dv = 0, \qquad \int_V F(v) dv = 1.$$

The turning rate $T_0[S]$ is bounded, and there exists a constant $\gamma = \gamma[S] > 0$ such that $T_0[S]/F \geq \gamma$ for all $(v, v') \in V \times V$, $x \in \mathbb{R}^n$, and $t > 0$.

LEMMA 2.2. *Let* $\zeta : \mathbb{R} \to \mathbb{R}$, $g : V \to \mathbb{R}$, *and let*

$$\phi_\epsilon^S[S] = \frac{T_\epsilon[S]F' + T_\epsilon^*[S]F}{2}, \qquad \phi_\epsilon^A[S] = \frac{T_\epsilon[S]F' - T_\epsilon^*[S]F}{2}$$

denote, respectively, the symmetric and antisymmetric parts of $T_\epsilon[S]F'$. *Then*

$$\int_V \int_V \mathcal{T}_\epsilon(Fg)\zeta(g)dv = \frac{1}{2}\int_V \int_V \phi_\epsilon^S[S](g - g')(\zeta(g) - \zeta(g'))dv'dv$$

$$+ \frac{1}{2}\int_V \int_V \phi_\epsilon^A[S](g + g')(\zeta(g) - \zeta(g'))dv'dv.$$

The same holds for $\mathcal{T}_\epsilon[S]$ *with analogous definitions of* $\phi_\epsilon^S[S]$ *and* $\phi_\epsilon^A[S]$.

Proof. See Lemma 1 in [3] for the proof.   □

With $g = f/F$ and $\zeta = \mathrm{id}$ one obtains the following.

LEMMA 2.3. *Let Assumption* 2.1 *hold. Then the entropy equality*

$$\int_V \mathcal{T}_0[S](f)\frac{f}{F}dv = \frac{1}{2}\int_V\int_V \phi_0^S[S]\left(\frac{f}{F} - \frac{f'}{F'}\right)^2 dv'dv \geq 0$$

holds. For $g \in L^2(V; dv/F)$*, the equation* $\mathcal{T}_0[S](f) = g$ *has a unique solution* $f \in L^2(V; dv/F)$ *satisfying* $\int_V f dv = 0$ *if and only if* $\int_V g dv = 0$*.*

Proof. See Lemma 2 in [3] for the proof.   □

From the entropy equality, we deduce that

$$f_0(x, v, t) = \rho_0(x, t)F(v).$$

Since $\mathcal{T}_{0S}[S_0, S_1](f_0) = 0$, we obtain

$$\mathcal{T}_0[S](f_1) = -vF \cdot \nabla\rho_0 - \rho_0\mathcal{T}_1[S_0](F).$$

The right-hand side satisfies the solvability condition from Lemma 2.3, and therefore the solution can be written as

$$f_1 = -\kappa(x, v, t) \cdot \nabla\rho_0(x, t) - \Theta(x, v, t)\rho_0(x, t) + \rho_1(x, t)F(v),$$

where $\kappa = \kappa[S_0]$ and $\Theta = \Theta[S_0]$ are the solutions of

$$\mathcal{T}_0[S_0](\kappa) = vF, \qquad \mathcal{T}_0[S_0](\Theta) = \mathcal{T}_1[S_0](F),$$

and ρ_1 is the macroscopic density of f_1, which is a new unknown. By passing to the limit $\epsilon \to 0$ in (2.3), the convection-diffusion equation reads

$$\partial_t\rho_0 - \nabla \cdot (D[S_0]\nabla\rho_0 - \rho_0 H[S_0]) = 0,$$

where

$$D[S_0](x, t) = \int_V v \otimes \kappa[S_0](x, v, t)dv, \qquad H[S_0] = -\int_V v\Theta[S_0](x, v, t)dv,$$

together with

$$\frac{\partial S_0}{\partial t} = \Delta S_0 + \rho_0 - S_0.$$

The specific form of $D[S_0]$ and $H[S_0]$ will depend on the choice of the turning kernels and will be discussed later.

3. Global solution of the kinetic model. In this section we show that solutions of the coupled system (1.6)–(1.9) in two and three dimensions do not blow up in finite time for fixed $\epsilon > 0$ if the turning kernel satisfies a certain structure condition. Without loss of generality we set $\epsilon = 1$ in (1.6) and $\alpha = 1$ in (1.8). We consider two problems, namely, the elliptic and the parabolic equations for the chemo-attractant.

We start with an inequality of Gronwall type in the next lemma. Since it is of the nonstandard form among the Gronwall-type inequalities, we present its proof for clarity, although the proof is similar to that of the usual one.

LEMMA 3.1. *Let a and b be positive constants. Let $y(t)$ and $y'(t)$ be positive and differentiable in t and satisfy*

(3.1) $$y' \leq ay \ln y' + by.$$

Then

$$y(t) \leq \left[y(0) \exp \left(2b \int_0^t e^{-2as} ds \right) \right]^{\exp(2at)}.$$

Proof. We subtract and add $\ln y$ from the right-hand side of (3.1) to get $y' \leq ay \ln y + ay \ln (\ln y)' + by$. Dividing both sides of the above inequality by y, we get $(\ln y)' \leq a \ln y + a \ln (\ln y)' + b$. Set $z = \ln y$ to get $z' \leq az + a \ln z' + b$. Since we may assume $\ln z' \leq (1/2a)z'$ (otherwise, $\ln z' \leq C$ and the above inequality reduces to a standard Gronwall inequality), we have $z' \leq az + \frac{1}{2}z' + b$. We get $z' \leq 2az + 2b$, where $z = \ln y$. Using a standard Gronwall argument, we deduce the lemma. □

The structure condition on the turning kernel $T[S]$ is assumed to be as follows.

Assumption 3.2 (structure condition). There exist nonnegative constants $C_i \geq 0$, $i = 1, 2, \ldots, 5$, such that for all $x \in \mathbb{R}^n$, $n = 2, 3$, $v, v' \in V$, $t \in \mathbb{R}^+$, and $S \in W^{1,\infty}(\mathbb{R}^n)$, the turning kernel T satisfies

(3.2) $$\begin{aligned} 0 \leq T_\epsilon[S](x, v, v', t) &\leq C_1 + C_2 S(x + \epsilon v, t) + C_3 S(x - \epsilon v', t) \\ &\quad + C_4 |\nabla S(x + \epsilon v, t)| + C_5 |\nabla S(x - \epsilon v', t)|, \end{aligned}$$

(3.3) $$\begin{aligned} |\nabla T_\epsilon[S](x, v, v', t)| &\leq C_2 |\nabla S(x + \epsilon v, t)| + C_3 |\nabla S(x - \epsilon v', t)| \\ &\quad + C_4 |\nabla^2 S(x + \epsilon v, t)| + C_5 |\nabla^2 S(x - \epsilon v', t)|. \end{aligned}$$

This means that the cells can measure the concentration and the spatial gradient of the chemo-attractant up to a distance ϵ from their position, and this may affect the movement of the cells.

Remark 3.3. The turning kernel, as given above, describes the turning from direction v' into direction v. This means that the actual or "old" direction is evaluated by checking backwards, whereas the evaluation of possible new directions are checked forwards (e.g., by lamelliopodial protrusion). Checking the possible new directions backwards if compared to the actual direction of motion is also possible and could have been taken into account in the following considerations. Nevertheless, it is important to note that a forward evaluation of the actual direction v' causes a technical problem in our approach so far.

We first consider the case that the chemo-attractant equation is of elliptic type.

3.1. Elliptic case: $\tau = 0$. In this part, we consider the elliptic equation for the chemo-attractant S for two cases: $\beta > 0$ and $\beta = 0$. When $\beta > 0$ we may set $\beta = 1$ without loss of generality. So

(3.4) $$-\Delta S = \rho - \beta S, \quad \beta \in \{0, 1\}, \quad n = 2, 3.$$

For $n = 2$ we need some preliminaries and start with elementary properties of the Bessel potential G in two dimensions.

LEMMA 3.4. *Let G be the Bessel potential in $\mathbb{R}^2$. Then $G \in L^p(\mathbb{R}^2)$ for any p with $1 \leq p < \infty$ and $\nabla G \in L^p(\mathbb{R}^2)$ for any p with $1 \leq p < 2$. Furthermore,*

(3.5) $$\|G\|_{L^p(\mathbb{R}^2)} \leq Cp, \quad 1 \leq p < \infty,$$

(3.6) $$\|\nabla G\|_{L^p(\mathbb{R}^2)} \leq C \frac{2p}{2 - p}, \quad 1 \leq p < 2.$$

Proof. For $n = 2$, the Bessel potential is (cf. (2.1))

$$G(x) = \frac{1}{4\pi} \int_0^\infty e^{-\pi \frac{|x|^2}{4s} - \frac{s}{4\pi}} \frac{ds}{s}.$$

Using a change of variables, we have

$$\|G\|_{L^p(\mathbb{R}^2)} \leq C \int_0^\infty \frac{e^{-s}}{s} \big\| e^{-\frac{|x|^2}{4s}} \big\|_{L^p(\mathbb{R}^2)} ds \leq C \int_0^\infty e^{-s} s^{-1+1/p} ds \leq Cp.$$

We thus obtain (3.5). In a similar way we get

$$\|\nabla G\|_{L^p(\mathbb{R}^2)}^p \leq C \int_0^\infty \frac{e^{-s}}{s^2} \big\| x e^{-\frac{|x|^2}{4s}} \big\|_{L^p(\mathbb{R}^2)} ds \leq C \int_0^\infty e^{-s} s^{-\frac{3}{2}+\frac{1}{p}} ds \leq C \frac{2p}{2-p},$$

as long as $1 \leq p < 2$. Therefore we deduce (3.6). $\square$

The next lemma shows various estimates for the chemo-attractant S.

LEMMA 3.5. *Let S be a solution of* (3.4) *in $\mathbb{R}^2$. Then S satisfies the following estimates:*

$$(3.7) \quad \|S(t)\|_{L^p(\mathbb{R}^2)} + \|\nabla S(t)\|_{L^q(\mathbb{R}^2)} \leq C(p,q)\|\rho_0\|_{L^1(\mathbb{R}^2)}, \quad 1 \leq p < \infty, \ 1 \leq q < 2,$$

$$(3.8) \qquad \|\nabla S(t)\|_{L^2(\mathbb{R}^2)} \leq C\|\rho_0\|_{L^1(\mathbb{R}^2)} \big[\ln \big(\|\rho(t)\|_{L^2(\mathbb{R}^2)}^2 + 1 \big) \big]^{1/2}.$$

Proof. The first estimate (3.7) is an easy consequence of mass conservation, Lemma 3.4, and Young's inequality (see, e.g., [7, pp. 624–625]). Thus it suffices to show the estimate (3.8).

From (3.4) we obtain the Fourier transform $\hat{S}(\xi) = \hat{\rho}(\xi)/(|\xi|^2 + 1)$, and thus

$$\|\nabla S(t)\|_{L^2(\mathbb{R}^2)} = \|\xi \hat{S}(t)\|_{L^2(\mathbb{R}^2)} = \left\| \frac{|\xi|\hat{\rho}(t)}{|\xi|^2 + 1} \right\|_{L^2(\mathbb{R}^2)},$$

where Plancherel's equality is used. The above integral can be estimated by splitting $\mathbb{R}^2$ of the ξ-space into two parts:

$$\int_{\mathbb{R}^2} \frac{|\xi \hat{\rho}(t)|^2}{(|\xi|^2 + 1)^2} d\xi = \int_{|\xi|<R} + \cdots + \int_{|\xi|>R} + \cdots = I_1 + I_2,$$

where $R > 0$ will be chosen later. Using Hölder's inequality and Plancherel's equality we have

$$I_1 \leq \|\hat{\rho}(t)\|_{L^\infty(\mathbb{R}^2)}^2 \int_{|\xi|<R} \frac{|\xi|^2}{(|\xi|^2 + 1)^2} d\xi \leq C\|\rho(t)\|_{L^1(\mathbb{R}^2)}^2 \ln(R^2 + 1),$$

$$I_2 \leq \left\| \frac{|\xi|}{|\xi|^2 + 1} \right\|_{L^\infty(|\xi|>R)}^2 \|\hat{\rho}(t)\|_{L^2(\mathbb{R}^2)}^2 \leq C R^{-2} \|\rho(t)\|_{L^2(\mathbb{R}^2)}^2.$$

Therefore, by choosing $R = \|\rho(t)\|_{L^2(\mathbb{R}^2)}$, we obtain

$$\|\nabla S(t)\|_{L^2(\mathbb{R}^2)} \leq C\|\rho(t)\|_{L^1(\mathbb{R}^2)} \{\ln(R^2 + 1)\}^{1/2} + CR^{-1}\|\rho(t)\|_{L^2(\mathbb{R}^2)}$$

$$\leq C \left[1 + \|\rho(t)\|_{L^1(\mathbb{R}^2)} \big\{ \ln \big(\|\rho(t)\|_{L^2(\mathbb{R}^2)}^2 + 1 \big) \big\}^{1/2} \right].$$

Since $\|\rho\|_{L^1(\mathbb{R}^2)} = \|f_0\|_{L^1(\mathbb{R}^2 \times V)}$, we deduce (3.8) and our lemma. $\square$

The next theorem shows global existence of solutions for system (1.6)–(1.9) with $\tau = 0$, namely, blowup does not happen in finite time.

THEOREM 3.6. *Suppose the chemo-attractant equation is of elliptic type ($\tau = 0$). Assume that $f_0, \nabla f_0 \in (L^1 \cap L^\infty)(\mathbb{R}^n \times V)$, with $n = 2, 3$.*

1. *Case $n = 2$, $\beta > 0$: Let Assumption* 3.2 *hold. Then there exist global solutions* $f, \nabla f \in L^\infty_{\text{loc}}((0, \infty); L^1 \cap L^\infty(\mathbb{R}^2 \times V))$ *and* $S \in L^\infty_{\text{loc}}((0, \infty); W^{1,p}(\mathbb{R}^2))$ *for all* $1 \leq p \leq +\infty$ *of the system* (1.6)–(1.9) *with* $\epsilon > 0$ *fixed but arbitrary.*

2. *Case $n = 2$, $\beta = 0$: Let Assumption* 3.2 *hold with* $C_2 = C_3 = C_5 = 0$. *Then there exist global solutions* $f, \nabla f \in L^\infty_{\text{loc}}((0, \infty); L^1 \cap L^\infty(\mathbb{R}^2 \times V))$ *and* $\nabla S \in L^\infty_{\text{loc}}((0, \infty); L^p(\mathbb{R}^2))$ *for all* $2 < p \leq \infty$ *of the system* (1.6)–(1.9) *with* $\epsilon > 0$ *fixed but arbitrary.*

3. *Case $n = 3$, $\beta > 0$: Let Assumption* 3.2 *hold with* $C_3 = C_5 = 0$. *Then there exist global solutions* $f, \nabla f \in L^\infty_{\text{loc}}((0, \infty); L^1 \cap L^\infty(\mathbb{R}^3 \times V))$ *and* $S \in L^\infty_{\text{loc}}((0, \infty);$ $W^{1,p}(\mathbb{R}^3))$ *for all* $1 \leq p \leq +\infty$ *of the system* (1.6)–(1.9) *with* $\epsilon > 0$ *fixed but arbitrary.*

4. *Case $n = 3$, $\beta = 0$: Let Assumption* 3.2 *hold with* $C_3 = C_5 = 0$. *Then there exist global solutions* $f, \nabla f \in L^\infty_{\text{loc}}((0, \infty); L^1 \cap L^\infty(\mathbb{R}^3 \times V))$ *and* $S \in L^\infty_{\text{loc}}((0, \infty);$ $L^p(\mathbb{R}^3))$ *for any* $3 < p \leq \infty$ *and* $\nabla S \in L^\infty_{\text{loc}}((0, \infty); L^p(\mathbb{R}^3))$ *for any* $3/2 < p \leq \infty$ *of the system* (1.6)–(1.9) *with* $\epsilon > 0$ *fixed but arbitrary.*

Proof. (a) We first consider the case $n = 2$ and $\beta > 0$. Without loss of generality, we assume $\epsilon = 1$. Mass is conserved for ρ, and thus $\|\rho(\cdot, t)\|_{L^1(\mathbb{R}^2)} = \|f_0\|_{L^1(\mathbb{R}^2 \times V)}$.

$$\partial_t f(x, v, t) + v \cdot \nabla_x f(x, v, t) = \int_V T[S](x, v, v', t) f(x, v', t) dv'$$
$$- \int_V T[S](x, v', v, t) f(x, v, t) dv'.$$

Using Assumption 3.2, we get

$$f(x, v, t) \leq f_0(x - vt, v) + C \int_0^t \rho(x - vs, t - s) ds + C f_1(x, v, t) + C f_2(x, v, t),$$

where f_1 and f_2 satisfy

$$\partial_t f_1(x, v, t) + v \cdot \nabla_x f_1(x, v, t) = \int_V [S(x + v, t) + |\nabla S(x + v, t)|] f(x, v', t) dv',$$
$$\partial_t f_2(x, v, t) + v \cdot \nabla_x f_2(x, v, t) = \int_V [S(x - v', t) + |\nabla S(x - v', t)|] f(x, v', t) dv',$$

with initial conditions $f_i(x, v, 0) = 0$ for $i = 1, 2$. We first consider f_1. One can easily see that

$$f_1(x, v, t) = \int_0^t [S(x - vs + v, t - s) + |\nabla S(x - vs + v, t - s)|] \rho(x - vs, t - s) ds.$$

After simple calculations, we obtain the following estimates:

$$\|f_1(\cdot, \cdot, t)\|_{L^p(\mathbb{R}^2 \times V)} \leq C \sup_{0 \leq s \leq t} \|S(\cdot, s)\|_{W^{1,p}(\mathbb{R}^2)} \int_0^t \|\rho(\cdot, t - s)\|_{L^p(\mathbb{R}^2)} ds.$$

For the term f_2, we have

$$f_2(x, v, t) = \int_0^t \int_V [S(x - vs - v', t - s) + |\nabla S(x - vs - v', t - s)| f(x - vs, v', t - s) dv' ds.$$

Applying Young's inequality, we get

$$\|(S(\cdot, t - s) + |\nabla S(\cdot, t - s)|) * f(x - vs, \cdot, t - s)\|_{L^\infty(V)}$$
$$\leq \sup_{0 < s < t} \|S(\cdot, s)\|_{W^{1,p}(\mathbb{R}^2)} \|f(x - vs, \cdot, t - s)\|_{L^{p'}(V)},$$

where p and p' are conjugate exponents. If $p \geq 2$, then $p' \leq p$, and so we have, by interpolation between p and 1,

$$\|f(x - vs, \cdot, t - s)\|_{L^{p'}(V)} \leq C(V)\|f(x - vs, \cdot, t - s)\|_{L^p(V)}.$$

Hence,

$$\|f_2(\cdot, \cdot, t)\|_{L^p(\mathbb{R}^2 \times V)} \leq \sup_{0 < s < t} \|S(\cdot, s)\|_{W^{1,p}(\mathbb{R}^2)} \int_0^t \|f(\cdot, \cdot, t - s)\|_{L^p(\mathbb{R}^2 \times V)} ds.$$

Therefore, summing up the estimates above, we obtain for $p \geq 2$

$$\begin{aligned} \|f(\cdot, \cdot, t)\|_{L^p(\mathbb{R}^2 \times V)} &\leq \|f_0(\cdot, \cdot)\|_{L^p(\mathbb{R}^2 \times V)} \\ (3.9) \qquad &+ C\left(1 + \sup_{0 \leq s \leq t} \|S(\cdot, s)\|_{W^{1,p}(\mathbb{R}^2)}\right) \int_0^t \|f(\cdot, \cdot, s)\|_{L^p(\mathbb{R}^2 \times V)}. \end{aligned}$$

By Lemma 3.5, we have for $p = 2$

$$\begin{aligned} \|f(\cdot, \cdot, t)\|_{L^2(\mathbb{R}^2 \times V)} &\leq \|f_0(\cdot, \cdot)\|_{L^2(\mathbb{R}^2 \times V)} \\ &+ C\left(1 + \sup_{0 \leq s \leq t} \left[\ln\left(\|f\|_{L^2(\mathbb{R}^2 \times V)}^2 + 1\right)\right]^{1/2}\right) \int_0^t \|f(\cdot, \cdot, s)\|_{L^2(\mathbb{R}^2 \times V)}. \end{aligned}$$

Then, applying Gronwall's inequality as in Lemma 3.1, we obtain $f \in L^2(\mathbb{R}^2 \times V)$. Now, using bootstrap arguments we obtain the L^∞-estimate by applying repeatedly Lemma 3.4, Young's inequality, and Gronwall's inequality. Next we show L^∞-estimates for the derivatives of f. For convenience let $j = 1, 2$ be arbitrary but fixed, and we denote by $\tilde{f}$ and $\tilde{T}[S]$ the partial derivatives $\partial_{x_j} f$ and $\partial_{x_j} T[S]$, respectively.

$$\begin{aligned} \partial_t \tilde{f}(x, v, t) + v \cdot \nabla_x \tilde{f}(x, v, t) &= \int_V \tilde{T}[S](x, v, v', t) f(x, v', t) dv' \\ &+ \int_V T[S](x, v, v', t) \tilde{f}(x, v', t) dv' \\ &- \int_V \tilde{T}[S](x, v', v, t) f(x, v, t) dv' \\ &- \int_V T[S](x, v', v, t) \tilde{f}(x, v, t) dv'. \end{aligned}$$

Then, in the same manner as before, we obtain

$$\tilde{f}(x, v, t) \leq \tilde{f}_0(x - vt, v) + C\tilde{f}_1(x, v, t) + C\tilde{f}_2(x, v, t) + C\tilde{f}_3(x, v, t) + C\tilde{f}_4(x, v, t),$$

where

$$\begin{aligned} \tilde{f}_1(x, v, t) &= \int_0^t \int_V \tilde{T}[S](x - vs, v, v', t - s) f(x - vs, v', t - s) dv' ds, \\ \tilde{f}_2(x, v, t) &= \int_0^t \int_V T[S](x - vs, v, v', t - s) \tilde{f}(x - vs, v', t - s) dv' ds, \\ \tilde{f}_3(x, v, t) &= -\int_0^t \int_V \tilde{T}[S](x - vs, v', v, t - s) f(x - vs, v, t - s) dv' ds, \\ \tilde{f}_4(x, v, t) &= -\int_0^t \int_V T[S](x - vs, v', v, t - s) \tilde{f}(x - vs, v, t - s) dv' ds. \end{aligned}$$

We consider first $\tilde{f}_1(x, v, t)$. Here we use the fact that the L^∞- and L^p-norms of f, depending on t, are bounded, which was shown above. Therefore we have

$$|\tilde{f}_1(x, v, t)| \leq \sup_{0 < s < t} \|f(\cdot, s)\|_{L^\infty(\mathbb{R}^2 \times V)} \int_0^t \int_V |\tilde{T}[S](x - vs, v, v', t - s)| dv' ds.$$

Using Assumption 3.2, one can easily see

$$\|\tilde{f}_1(\cdot, \cdot, t)\|_{L^p(\mathbb{R}^2 \times V)} \leq C \sup_{0 < s < t} \|f(\cdot, s)\|_{L^\infty(\mathbb{R}^2 \times V)} \sup_{0 < s < t} \|S(\cdot, s)\|_{W^{2, p}(\mathbb{R}^2)}$$
$$\leq C \sup_{0 < s < t} \|f(\cdot, s)\|_{L^\infty(\mathbb{R}^2 \times V)} \sup_{0 < s < t} \|\rho(\cdot, s)\|_{L^p(\mathbb{R}^2)} \leq C = C(t, |V|),$$

where we used a standard estimate for the chemo-attractant equation. Since $\tilde{f}_3$ has the same structure as $\tilde{f}_1$, $\tilde{f}_3$ satisfies the estimates above. On the other hand, $\tilde{f}_2$ is estimated, due to Assumption 3.2, as follows:

$$|\tilde{f}_2(x, v, t)| \leq \sup_{0 < s < t} \|S(\cdot, s)\|_{W^{1, \infty}(\mathbb{R}^2)} \int_0^t \int_V \tilde{f}(x - vs, v', t - s) dv' ds.$$

Again, due to a standard estimate for the chemo-attractant equation, we get

$$|\tilde{f}_2(x, v, t)| \leq \sup_{0 < s < t} \|\tilde{f}\|_{L^q(\mathbb{R}^2)} \int_0^t \int_V \tilde{f}(x - vs, v', t - s) dv' ds,$$

where q is sufficiently large (i.e., $q > 2$). Integration over $\mathbb{R}^2 \times V$ yields

$$\|\tilde{f}_2(\cdot, \cdot, t)\|_{L^p(\mathbb{R}^2 \times V)} \leq C \int_0^t \|\tilde{f}(\cdot, \cdot, t - s)\|_{L^p(\mathbb{R}^2)} ds,$$

where we again used the boundedness of the L^p-norm of f and $C = C(|V|, t)$. $\tilde{f}_4$ can be treated in the same manner, so we omit the details. To sum up, we obtain

$$\|\nabla f(\cdot, \cdot, t)\|_{L^p(\mathbb{R}^2 \times V)} \leq C(|V|, t) + C(|V|, t) \int_0^t \|\nabla f(\cdot, \cdot, t - s)\|_{L^p(\mathbb{R}^2 \times V)} ds.$$

Gronwall's inequality justifies our claim. Repeating this process for higher regularity of f and S, we can easily see that this estimate is valid also in case $p = \infty$. This completes the proof of the case $\beta > 0$.

(b) Next we consider the case $n = 2$, $\beta = 0$. Again, for simplicity, we assume $\epsilon = 1$. We first decompose ∇S into two parts,

$$\nabla S = \nabla S^L + \nabla S^S = \rho * \left(-\frac{x}{2\pi |x|^2} \mathbf{I}_{|x| \geq 1} \right) + \rho * \left(-\frac{x}{2\pi |x|^2} \mathbf{I}_{|x| \leq 1} \right),$$

where $\mathbf{I_A}$ denotes the characteristic function of a set A. By mass conservation and Young's inequality, we have

$$\|\nabla S^L(t)\|_{L^\infty(\mathbb{R}^2)} \leq \frac{1}{2\pi} \|f_0\|_{L^1(\mathbb{R}^2 \times V)}.$$

Hence the estimate reduces to considering ∇S^S only, and we may replace ∇S by ∇S^S in the assumption on the turning kernel. Following similar procedures to those described in the case $\beta > 0$, we obtain for $p \geq 1$

$$f(x, v, t) \leq f_0(x - vt, v) + C \int_0^t \rho(x - vs, t - s) ds + C f_1(x, v, t),$$

where

$$f_1(x, v, t) = \int_0^t |\nabla S^S(x - vs + v, t - s)| \rho(x - vs, t - s) ds.$$

Simple calculations show

$$\|f_1(\cdot, \cdot, t)\|_{L^p(\mathbb{R}^2 \times V)} \le C \sup_{0 \le s \le t} \|\nabla S^S(\cdot, s)\|_{L^p(\mathbb{R}^2)} \int_0^t \|\rho(\cdot, t - s)\|_{L^p(\mathbb{R}^2)} ds.$$

To sum up, we obtain

$$\|f(\cdot, \cdot, t)\|_{L^p(\mathbb{R}^2 \times V)} \le C + C \left(1 + \sup_{0 \le s \le t} \|\nabla S^S(\cdot, s)\|_{L^p(\mathbb{R}^2)} \right)$$

(3.10)
$$\times \int_0^t \|f(\cdot, \cdot, t - s)\|_{L^p(\mathbb{R}^2 \times V)} ds.$$

Here we note that the above a priori estimate (3.10) holds for all $p \ge 1$. First we choose a specific p with $1 < p < 2$, which ensures, due to Young's inequality, that

$$\|\nabla S^S(\cdot, t)\|_{L^p(\mathbb{R}^2)} \le C \|f_0\|_{L^1(\mathbb{R}^2 \times V)}.$$

Then by Gronwall's inequality we get a bound, globally in time, for f in $L^p(\mathbb{R}^2)$ for such chosen p. By bootstrap arguments, we obtain $f \in L^\infty_{\text{loc}}([0, \infty); L^\infty(\mathbb{R}^2 \times V))$.

By similar procedures to those given in the proof of Theorem 3.6, an L^∞-estimate for ∇f can be obtained. $\nabla S \in L^\infty((0, \infty); L^p(\mathbb{R}^2))$, $2 < p \le \infty$, is due to the Hardy–Littlewood–Sobolev theorem (see [20, pp. 119–120]). Since this is also verified by embedding arguments for general elliptic equations, we skip the details.

Remark 3.7. Although similar results, in the theorem above, are expected for nonzero C_2, C_3, C_5 also in case $\beta = 0$, there are some technical difficulties in proving global existence when the chemo-attractant equation is of elliptic type. Indeed, the chemo-attractant equation becomes the Poisson equation without decay term $-\Delta S = \rho$, and thus S has the Newtonian potential representation, i.e., $S = \Gamma * \rho$, where $\Gamma(x) = 1/2\pi \log |x|$. Due to the behavior of Γ at infinity, we cannot, in general, control S in terms of ρ. (We do not have these kind of estimates in Lemma 3.5 if $\beta = 0$.) Thus we leave the global existence as an open question for nonzero C_2, C_3, and C_5 in case $\beta = 0$ and $\tau = 0$.

(c) The three-dimensional case: In this situation, unlike the two-dimensional case in Theorem 3.6, it is not necessary to distinguish proofs for $\beta = 0$ and $\beta \ne 0$. We briefly explain why C_3, C_5 are assumed to be zero in three dimensions. Indeed, as seen in the previous calculations, we end up with the following estimate:

$$\|f(\cdot, \cdot, t)\|_{L^p(\mathbb{R}^3 \times V)} \le C + C \left(1 + \sup_{0 \le s \le t} \|S^S(\cdot, s)\|_{W^{1,p}(\mathbb{R}^3)} \right)$$

(3.11)
$$\times \int_0^t \|f(\cdot, \cdot, t - s)\|_{L^p(\mathbb{R}^3 \times V)} ds.$$

On the other hand, in three dimensions, due to behavior of the potential, we have

(3.12)
$$\|S^S(\cdot, s)\|_{W^{1,p}(\mathbb{R}^3)} \le C \|\rho_0\|_{L^1(\mathbb{R}^3)} \quad \text{for } 1 \le p < \frac{3}{2}.$$

However, in case C_3 or C_5 are nonzero, one can easily show that estimate (3.11) is still valid provided that $p \geq 2$ (compare the estimates for f_2 and f_4 before), but this does not enable us to use bootstrap arguments to get higher regularity for f because of (3.12). Therefore we assume $C_3 = C_5 = 0$. With this assumption the proof for the case $n = 3$ is similar to the case $n = 2$. □

Remark 3.8. It is worth mentioning that Theorem 3.6 also holds in case $n = 3$ when the turning kernel satisfies Assumption 3.2 with $C_2 = C_4 = 0$ instead of $C_3 = C_5 = 0$, namely,

$$0 \leq T[S](x, v, v', t) \leq C \left(1 + S(x - \epsilon v', t) + |\nabla S(x - \epsilon v', t)|\right),$$
$$|\nabla T[S](x, v, v', t)| \leq C(|\nabla S(x - \epsilon v', t)| + |\nabla^2 S(x - \epsilon v', t)|).$$

This can be seen by changing the roles of p and p' in the estimate of f_2 and by following a similar procedure to the one given for the proof of Theorem 3.6.

We do not know if the theorem above is also valid if the turning kernel fulfills the structure conditions (3.2) and (3.3) as in the two-dimensional case.

3.2. Parabolic case: $\tau > 0$. In this part, the parabolic equation for the chemo-attractant in (1.8) is considered. From now on we let $\tau = 1$ without loss of generality and, for simplicity, we set here $\alpha = 1$. Then (1.8) for S reads

$$(3.13) \qquad \partial_t S - \Delta S = \rho - \beta S, \quad S(x, 0) = S_0(x), \quad \beta \geq 0.$$

To make our arguments simpler, from now on we assume $S_0 = 0$ (compare Remark 3.11 in the following for the case $S_0 \neq 0$).

In the next lemma we recall some basic properties of Γ in two dimensions.

LEMMA 3.9. *Let Γ be the fundamental solution for the operator $\partial_t - \Delta_x + \beta$ in $\mathbb{R}^2$. Then $\Gamma \in L^p(\mathbb{R}^2)$ for any p with $1 \leq p < \infty$, and $\nabla \Gamma \in L^p(\mathbb{R}^2)$ for any q with $1 \leq p < 2$, satisfying*

$$\int_0^t \|\Gamma(\cdot, s)\|_{L^p(\mathbb{R}^2)} ds \leq C(\beta) p, \quad 1 \leq p < \infty,$$
$$\int_0^t \|\nabla \Gamma(\cdot, s)\|_{L^p(\mathbb{R}^2)} ds \leq C(\beta) \frac{2p}{2-p}, \quad 1 \leq p < 2.$$

Proof. The proof is similar to that of Lemma 3.4, so we omit details. □

In the next lemma, we show L^p- and L^2-estimates for S and ∇S, respectively.

LEMMA 3.10. *Let S be a solution of (3.13) in $\mathbb{R}^2$ and $S_0 = 0$. Then S satisfies the estimates*

$$(3.14) \qquad \|S(t)\|_{L^p(\mathbb{R}^2)} + \|\nabla S(t)\|_{L^q(\mathbb{R}^2)} \leq C(\beta, p, q)\|\rho_0\|_{L^1(\mathbb{R}^2)},$$

where $1 \leq p < \infty$, $1 \leq q < 2$, and

$$\|\nabla S(t)\|_{L^2(\mathbb{R}^2)}^2 \leq C \left(1 + \|\rho_0\|_{L^1(\mathbb{R}^2)} \left(1 + (\ln t)_+ + \sup_{0 \leq \tau \leq t} \left| \ln \|\rho(\tau)\|_{L^2(\mathbb{R}^2)}^2 \right| \right) \right),$$

(3.15)

where $(f)_+$ indicates the positive part of f.

Proof. By Duhamel's principle and using the fundamental solution Γ in (2.2), we have

$$(3.16) \qquad S(x, t) = \int_0^t \Gamma(\cdot, s) * \rho(\cdot, t - s) ds.$$

By using Lemma 3.9, mass conservation, and Young's inequality, we easily get (3.14). To estimate $\|\nabla S\|_{L^2(\mathbb{R}^2)}$, we take the Fourier transform of (3.16) and use Plancherel's equality to get

$$\|\nabla S(t)\|_{L^2(\mathbb{R}^2)} = \|\xi \hat{S}(t)\|_{L^2(\mathbb{R}^2)} \leq \int_0^t \||\xi|\hat{\Gamma}(\cdot,s)\hat{\rho}(\cdot,t-s)\|_{L^2(\mathbb{R}^2)}ds = \int_0^r + \cdots + \int_r^t + \cdots,$$

where $r > 0$ will be chosen appropriately later. Note that the Fourier transform of Γ is $\hat{\Gamma}(\xi,s) = \exp(-s(4\xi^2 + \beta))$. For $0 < s < r$, due to the Hölder's inequality and Plancherel's equality, we have

$$\int_0^r \cdots \leq \int_0^r \||\xi|\exp(-s(4\xi^2 + \beta))\|_{L^\infty(\mathbb{R}^2)}\|\hat{\rho}(s)\|_{L^2(\mathbb{R}^2)}ds$$

$$\leq C \sup_{0 \leq s \leq t} \|\rho\|_{L^2(\mathbb{R}^2)} \int_0^r s^{-1/2}ds \leq Cr^{1/2} \sup_{0 \leq s \leq t} \|\rho\|_{L^2(\mathbb{R}^2)}.$$

For $r < s < t$, due to mass conservation and Hölder's inequality, now applied in the opposite way, we have

$$\int_r^t \cdots \leq \int_r^t \||\xi|\exp(-s(4\xi^2 + \beta))\|_{L^2(\mathbb{R}^2)}\|\hat{\rho}(s)\|_{L^\infty(\mathbb{R}^2)}ds$$

$$\leq C\|\rho_0\|_{L^1(\mathbb{R}^2)} \int_r^t \frac{1}{s}ds \leq C\|\rho_0\|_{L^1(\mathbb{R}^2)}|\ln t - \ln r|,$$

where we used $\|\hat{\rho}\|_{L^\infty(\mathbb{R}^2)} \leq \|\rho\|_{L^1(\mathbb{R}^2)}$. Therefore we obtain

$$\|\nabla S(t)\|_{L^2(\mathbb{R}^2)} \leq C\left(r^{1/2} \sup_{0 \leq s \leq t} \|\rho\|_{L^2(\mathbb{R}^2)} + \|\rho_0\|_{L^1(\mathbb{R}^2)}|\ln t - \ln r|\right).$$

By choosing $r = \min\{(\sup_{0 \leq s \leq t} \|\rho\|_{L^2(\mathbb{R}^2)})^{-2}, t\}$ in the above inequality, we deduce our lemma.   □

Remark 3.11. For the case $S_0 \neq 0$, which is assumed to be sufficiently smooth, one has

$$S(x,t) = \int_0^t \Gamma(\cdot,s) * \rho(\cdot,t-s)ds + \int_{\mathbb{R}^2} \Gamma(x-y,t)S_0(y)dy.$$

This gives the following variants of the estimates in the above lemma:

$$\|S(t)\|_{L^p(\mathbb{R}^2)} + \|\nabla S(t)\|_{L^q(\mathbb{R}^2)} \leq C\big(\|S_0\|_{L^p(\mathbb{R}^2)} + \|\nabla S_0\|_{L^q(\mathbb{R}^2)} + \|\rho_0\|_{L^1(\mathbb{R}^2)}\big),$$

where $1 \leq p < \infty$, $1 \leq q < 2$ and

$$\|\nabla S(t)\|_{L^2(\mathbb{R}^2)}^2 \leq C\left(1 + \|\nabla S_0\|_{L^2(\mathbb{R}^2)}\right.$$

$$\left. + \|\rho_0\|_{L^1(\mathbb{R}^2)}\left(1 + (\ln t)_+ + \sup_{0 \leq \tau \leq t}\big|\ln\big(\|\rho(\tau)\|_{L^2(\mathbb{R}^2)}^2\big)\big|\right)\right).$$

Since computations are straightforward, we omit the details.

As in the previous elliptic case, we can establish global existence for the system (1.6)–(1.9) with $\tau = 1$. To be more precise, once we have the essential estimate (3.15) for $\|\nabla S\|_{L^2(\mathbb{R}^2)}$, its proof is more or less the same as that for the elliptic case with

$\tau = 0$. Regularity of S is due to standard theory of general parabolic equations. For dimension three, under the weaker assumptions of the turning kernel (Assumption 3.2 with $C_3 = C_5 = 0$) than those in the elliptic case, we can also show global existence of solutions. Since the arguments are straightforward if compared to the elliptic case, we just state the results and skip its proof.

THEOREM 3.12. *Suppose the chemo-attractant equation is of parabolic type. Assume that $f_0, \nabla f_0 \in (L^1 \cap L^\infty)(\mathbb{R}^n \times V)$.*

1. *(Case $n = 2$.) Let $\beta \geq 0$ and Assumption 3.2 hold. Then there exist global solutions $f, \nabla f \in L^\infty_{\mathrm{loc}}((0, \infty); (L^1 \cap L^\infty)(\mathbb{R}^2 \times V))$ and $S, \nabla S \in L^\infty_{\mathrm{loc}}((0, \infty); L^p(\mathbb{R}^2))$ for all $1 \leq p \leq +\infty$ of system (1.6)–(1.9).*

2. *(Case $n = 3$.) Let $\beta \geq 0$ and Assumption 3.2 with $C_3 = C_5 = 0$. Then there exist global solutions $f, \nabla f \in L^\infty_{\mathrm{loc}}((0, \infty); (L^1 \cap L^\infty)(\mathbb{R}^3 \times V))$ and $S, \nabla S \in L^\infty_{\mathrm{loc}}((0, \infty); L^p(\mathbb{R}^3))$ for all $1 \leq p \leq +\infty$ of system (1.6)–(1.9).*

4. Diffusion limits of the kinetic model. In this section, the diffusion limit for kinetic models of type (1.6)–(1.9) is presented. First, in a lemma, we review estimates for S which satisfies an equation of elliptic type, i.e.,

$$-\Delta S = \rho - \beta S, \quad \beta \geq 0, \ \ \text{in } \mathbb{R}^n, \ n = 2, 3.$$

We use standard arguments, which are known as potential theory. Proofs are straightforward (compare, e.g., [9, Chapters 2 and 8] and [20, Chapter V] for the two-dimensional case, and [16, Chapter 4] and [17, Chapters 4 and 6] for the three-dimensional case).

LEMMA 4.1. *Let $I = [0, T) \subset \mathbb{R}$ and $0 < T < \infty$. Suppose $\rho \in L^\infty(I; (W^{1,1}(\mathbb{R}^n) \cap W^{1,q}(\mathbb{R}^n)))$, where $q > n$. Let S satisfy the chemo-attractant equation of either elliptic or parabolic type with $\beta \geq 0$.*

(i) *In the case either $n = 2$, $\beta > 0$ or $n = 3$, $\beta \geq 0$, and S fullfils the chemo-attractant equation of either elliptic or parabolic type,*

$$S \in L^\infty(I; W^{2,p}(\mathbb{R}^n)) \cap L^\infty(I; \mathcal{C}^{2+\alpha}(\mathbb{R}^n)), \quad 1 \leq p < \infty, \ 0 < \alpha \leq \frac{q-n}{q},$$

and S satisfies the estimate

$$\|S\|_{L^\infty(I; W^{2,p}(\mathbb{R}^n))} + \|S\|_{L^\infty(I; \mathcal{C}^{2+\alpha}(\mathbb{R}^n))} \leq C \big(\|\rho\|_{L^\infty(I; W^{1,1}(\mathbb{R}^n))} + \|\rho\|_{L^\infty(I; W^{1,q}(\mathbb{R}^n))} \big).$$

(ii) *The result of (i) is true also for $n = 2$, $\beta = 0$, when S fullfils the chemo-attractant equation of parabolic type.*

(iii) *In the case $n = 2$ and $\beta = 0$ and S fullfils the chemo-attractant equation of elliptic type,*

$$\nabla S \in L^\infty(I; W^{1,p}(\mathbb{R}^2)) \cap L^\infty(I; \mathcal{C}^{1+\alpha}(\mathbb{R}^2)), \quad 1 \leq p < \infty, \ 0 < \alpha \leq \frac{q-2}{q},$$

and S satisfies the estimate

$$\|\nabla S\|_{L^\infty(I; W^{1,p}(\mathbb{R}^2))} + \|\nabla S\|_{L^\infty(I; \mathcal{C}^{1+\alpha}(\mathbb{R}^2))} \leq C \big(\|\rho\|_{L^\infty(I; W^{1,1}(\mathbb{R}^2))} + \|\rho\|_{L^\infty(I; W^{1,q}(\mathbb{R}^2))} \big).$$

As in [3] we need similar assumptions on $\phi^S_\epsilon[S]$ and $\phi^A_\epsilon[S]$, which are the symmetric and antisymmetric parts of $T_\epsilon[S]$ (see Lemma 2.2).

Assumption 4.2. There exist $\gamma > 0$ and a nondecreasing function $\Lambda \in L^\infty_{\text{loc}}$, such that

$$\phi^S_\epsilon[S] \geq \gamma \left(1 - \epsilon \Lambda \left(\|\nabla S\|_{W^{1,\infty}(\mathbb{R}^n)}\right)\right) FF',$$

$$\int_V \frac{\phi^A_\epsilon[S]^2}{F\phi^S_\epsilon[S]} dv' \leq \epsilon^2 \Lambda \left(\|\nabla S\|_{W^{1,\infty}(\mathbb{R}^n)}\right),$$

where $F \in L^\infty(V)$ is a positive velocity distribution satisfying Assumption 2.1.

THEOREM 4.3. *Let Assumptions* 2.1 *and* 4.2 *hold and let* $q > n$ *with* $n = 2, 3$. *Suppose that the equation for the chemo-attractant S is either of elliptic ($\tau = 0$) or parabolic type ($\tau \neq 0$). Let one of following conditions hold:*

(i) *If* $\tau = 0$, $n = 2$, $\beta > 0$ *or if* $\tau > 0$, $n = 2$, $\beta \geq 0$, *the turning kernel satisfies Assumption* 3.2.

(ii) *If* $\tau = 0$, $n = 2$, $\beta = 0$, *the turning kernel satisfies Assumption* 3.2 *with* $C_2 = C_3 = C_5 = 0$.

(iii) *If* $\tau \geq 0$, $n = 3$, $\beta \geq 0$, *the turning kernel satisfies Assumption* 3.2 *with* $C_3 = C_5 = 0$.

Assume further that

$$f_0 \in \Upsilon_q \equiv W^{1,1}(\mathbb{R}^n \times V) \cap W^{1,q}\left(\mathbb{R}^n \times V; \frac{dxdv}{F^{q-1}}\right).$$

Then there exists $t^* > 0$, independent of ϵ, such that the solutions f_ϵ, S_ϵ satisfy

$$f_\epsilon \in L^\infty((0, t^*); \Upsilon_q),$$

$$\nabla S_\epsilon \in L^\infty((0, t^*); W^{1,p}(\mathbb{R}^n) \cap \mathcal{C}^{1+\alpha}(\mathbb{R}^n)), \quad 1 \leq p < \infty, \ \alpha = \frac{q-2}{q}$$

$$\textit{if } \tau = 0, \ n = 2, \ \beta = 0.$$

$$S_\epsilon \in L^\infty((0, t^*); W^{2,p}(\mathbb{R}^n) \cap \mathcal{C}^{2+\alpha}(\mathbb{R}^n)), \quad 1 \leq p < \infty, \ \alpha = \frac{q-n}{q} \textit{ in all other cases,}$$

$$(4.1) \qquad r_\epsilon = \frac{f_\epsilon - \rho_\epsilon F}{\epsilon} \in L^2\left((0, t^*); \mathbb{R}^n \times V : \frac{dxdvdt}{F}\right).$$

Proof. This can be shown by following the same procedure as that given in the proof of Theorem 4 in [3], and therefore we present only a brief sketch of this proof. Simple calculations show

$$\frac{d}{dt} \int_{\mathbb{R}^n} \int_V \frac{f^q_\epsilon}{F^{q-1}} dvdx \leq C\Lambda\left(\|\nabla S\|_{W^{1,\infty}(\mathbb{R}^n)}\right) \int_{\mathbb{R}^n} \int_V \frac{f^q_\epsilon}{F^{q-1}} dvdx.$$

The next step is to estimate S_ϵ:

$$\|\nabla S_\epsilon(\cdot, t)\|_{\mathcal{C}^{1,\alpha}(\mathbb{R}^n)} \leq C\left(1 + \|\nabla \rho_\epsilon(\cdot, t)\|_{L^q(\mathbb{R}^n)}\right) \leq \tilde{C}\left(1 + \|\rho_\epsilon(\cdot, t)\|_{L^q(\mathbb{R}^n)}\right).$$

Here we used the estimates in Lemma 4.1.

$$\frac{d}{dt} \int_{\mathbb{R}^n} \int_V \frac{f^q_\epsilon}{F^{q-1}} dvdx \leq C\left[1 + \left(\int_{\mathbb{R}^n} \int_V \frac{f^q_\epsilon}{F^{q-1}} dvdx\right)^{\frac{1}{q}}\right] \int_{\mathbb{R}^n} \int_V \frac{f^q_\epsilon}{F^{q-1}} dvdx.$$

This shows the first two statements. The rest can be done by using the same method as that given in the proof of Theorem 4 in [3], and thus we omit the details. □

Now we are ready to prove the existence of the diffusion limit in a short time interval.

THEOREM 4.4. *Let the assumption of Theorem 4.3 hold. Suppose that the equation for the chemo-attractant S is of elliptic ($\tau = 0$) or parabolic ($\tau \neq 0$) type. Assume further that for families (S_ϵ), which are uniformly bounded in $L^\infty_{loc}([0,\infty); \mathcal{C}^{2+\alpha}(\mathbb{R}^n))$ for some α with $0 < \alpha \leq 1$, such that S_ϵ, ∇S_ϵ, and $\nabla^2 S_\epsilon$ converge to $S_0, \nabla S_0$, and $\nabla^2 S_0$ as $\epsilon \to 0$, respectively, in $L^p_{loc}([0,\infty); \mathbb{R}^n)$ for some $p > n/(n-1)$ with $n = 2, 3$, we have the convergence*

$$T_\epsilon[S_\epsilon] \to T_0[S_0] \quad in \ L^p_{loc}([0,\infty); \mathbb{R}^n \times \bar{V} \times \bar{V}),$$

$$(4.2) \qquad \frac{\mathcal{T}_\epsilon[S_\epsilon](F)}{\epsilon} = \frac{2}{\epsilon} \int_V \phi_\epsilon^A[S_\epsilon] dv' \to \mathcal{T}_1[S_0](F) \quad in \ L^p_{loc}([0,\infty); \mathbb{R}^n \times \bar{V}).$$

Then the solutions f_ϵ and S_ϵ of (1.6)–(1.9) satisfy

$$f_\epsilon \to \rho_0 F \quad in \ L^\infty((0, t^*); \Upsilon_q) \ weak \ *,$$

and for $\tau = 0$

$$\nabla S_\epsilon \to \nabla S_0 \quad in \ W^{1,q}_{loc}((0, t^*); \mathbb{R}^n), \quad 1 \leq q < \infty \ if \ n = 2, \ \beta = 0,$$
$$S_\epsilon \to S_0 \quad in \ W^{2,q}_{loc}((0, t^*); \mathbb{R}^n), \quad 1 \leq q < \infty \ otherwise,$$

whereas for $\tau \neq 0$

$$S_\epsilon \to S_0 \quad in \ L^q_{loc}((0, t^*); W^{2,q}(\mathbb{R}^n)), \quad 1 \leq q < \infty.$$

Proof. Since the proof is similar to that of Theorem 5 in [3], we again present only a brief sketch of the procedure. First we note, due to (4.1), that

$$J_\epsilon = \frac{1}{\epsilon} \int_V v f_\epsilon dv = \int_V v r_\epsilon dv \in L^2((0, t^*); L^2(\mathbb{R}^n))$$

uniformly in ϵ. From the cell conservation equation $\partial_t \rho_\epsilon + \operatorname{div} J_\epsilon = 0$, one can easily see that

$$\partial_t (\nabla S_\epsilon) \in L^2((0, t^*); L^2_{loc}(\mathbb{R}^n))$$

by considering the gradient of the convolution of (1.8). The strong convergence follows combining the above estimate and the parabolic regularity for the convolutions defining S_ϵ and ∇S_ϵ from ρ_ϵ. Therefore, the kinetic equation (1.7) leads to

$$\epsilon \frac{\partial f_\epsilon}{\partial t} + v \cdot \nabla_x f_\epsilon = -\rho_\epsilon \frac{\mathcal{T}[S_\epsilon](F)}{\epsilon} - \mathcal{T}_\epsilon[S_\epsilon](r_\epsilon).$$

By assumption (4.2) and passing to the limit, we obtain

$$\mathcal{T}_0[S_0](r_0) = -vF \cdot \nabla \rho_0 - \rho_0 \mathcal{T}_1[S_0](F).$$

This equation can be solved due to Lemma 2.3. The limit of the cell conservation equation is $\partial_t \rho_0 + \nabla \cdot J_0 = 0$ with $J_0 = \int_V v r_0 dv$. This completes the proof. $\quad\square$

5. Examples. When dealing with chemosensitive movement of biological species, questions of major interest are, How do the individuals "measure" the chemical signal? How is this information processed, and what kind of behavior results? The model we have introduced before and its macroscopic limit give a partial answer to this problem.

First we give a short summary of possible evaluations of the chemical signal by the cells as suggested by Tranquillo and Alt [22] and later discuss related examples. The individuals might evaluate the chemical signal

spatially - the signal is evaluated at (at least) two distinct locations around the individual, which are related to its direction (cf. Examples 5.1, 5.3, and 5.4);

temporal(ly) differential - the signal is evaluated at (at least) two different times (cf. Example 5.1);

positionally - the signal is evaluated momentarily (cf. Example 5.4);

directionally - the signal is evaluated along the individual direction or its relation to a directional signal field, e.g., a spatial gradient at its position (cf. Examples 5.1, 5.3, 5.4, 5.5, and 5.6).

Discussions of possible turning rates of the cells which depend on the given chemical signal in this context are also given in [1], [2], [18] and [10], [19].

In [10], [19] the macroscopic limit is formal. It is assumed that the turning kernel has an expansion in ϵ which is supposed to be given. Here the ϵ-expansion is directly related to possible evaluations of the chemo-attractant by the cells, and thus the connection between the micro- and macroparameters can be derived.

To understand the different influences of the evaluations of the chemical signal, our first example is very general and allows also dependencies on time derivatives of the chemo-attractant. Since we did not prove regularity for S_t so far, the macroscopic limit in this case has to be considered only formal. Nevertheless, from this example the other rigorous examples can be extracted later. Below we only consider the two-dimensional case, to keep the computations simple and since this case is the most interesting one biologically.

Example 5.1 (formal for $\alpha > 0$, rigorous for $\alpha = 0$). Let the turning kernel be of general type:

$$T_\epsilon[s] = \phi(S(x + \epsilon v, t), S(x - \epsilon v', t), S(x, t - \epsilon), \nabla S(x + \epsilon v, t), \nabla S(x - \epsilon v', t),$$

$$(5.1) \qquad \partial_t S(x + \epsilon v, t), \partial_t S(x - \epsilon v', t), \partial_t S(x, t - \epsilon), v) + \epsilon \psi \left(\frac{v \cdot v'}{|v||v'|} \right),$$

where $\phi : \mathbb{R}^{12} \to \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{R}$ are smooth and $\phi + \epsilon \psi$ is strictly positive (∇S contributes two entries, $\partial_{x_1} S$ and $\partial_{x_2} S$). Here S satisfies the chemo-attractant equation either of elliptic type or of parabolic type with $\alpha \geq 0$ and $\beta > 0$ in two dimensions. For $\alpha = 0$ the S-equation is completely decoupled. In this case the derivation given below is rigorous. We do not include direct dependencies such as $S(x, t), S_t(x, t), \nabla S(x, t)$ at this point. These will be discussed later.

We use the following notational abbreviations:

$$\phi[S, \nabla S, \partial_t S, v] := \phi(S(x, t), S(x, t), S(x, t), \nabla S(x, t),$$
$$\nabla S(x, t), \partial_t S(x, t), \partial_t S(x, t), \partial_t S(x, t), v),$$
$$\phi_i[S, \nabla S, \partial_t S, v] := \phi_i(S(x, t), S(x, t), S(x, t), \nabla S(x, t), \nabla S(x, t),$$
$$\partial_t S(x, t), \partial_t S(x, t), \partial_t S(x, t), v),$$

where $\phi_i(\cdots)$ indicates the partial derivative of ϕ with respect to the ith argument for $i = 1, 2, \ldots, 12$. By the asymptotic expansion of $T_\epsilon = T_0 + \epsilon T_1 + O(\epsilon^2)$, one can easily see that $T_0 = T_0[S, v] = \phi[S, \nabla S, \partial_t S, v]$ and

$$
\begin{aligned}
T_1 &= T_1[S, v, v'] \\
&= (\phi_1[S, \nabla S, \partial_t S, v]v - \phi_2[S, \nabla S, \partial_t S, v]v') \cdot \nabla S + \phi_3[S, \nabla S, \partial_t S, v]\partial_t S \\
&\quad + (\phi_{3+i}[S, \nabla S, \partial_t S, v]v - \phi_{5+i}[S, \nabla S, \partial_t S, v]v') \cdot \nabla S_{x_i} \\
&\quad + (\phi_8[S, \nabla S, \partial_t S, v]v - \phi_9[S, \nabla S, \partial_t S, v]v') \cdot \nabla S_t \\
&\quad - \phi_{10}[S, \nabla S, \partial_t S, v]\partial_t^2 S + \psi\left(\frac{v \cdot v'}{|v||v'|}\right),
\end{aligned}
$$

where we used the summation convention, which is understood over repeated indices running from 1 to 2. Furthermore, we define Φ, $\tilde{\Phi}$, $\hat{\Phi}$, and $\bar{\Phi}$ as follows:

$$
\Phi[S_0, \nabla S_0, \partial_t S_0] := \int_V T_0[S_0, v']dv', \qquad \tilde{\Phi}[S_0, \nabla S_0, \partial_t S_0, v] := \int_V T_1[S_0, v', v]dv',
$$

$$
\hat{\Phi}[S_0, \nabla S_0, \partial_t S_0, v] := \int_V T_1[S_0, v, v']f_0(v', x, t)dv',
$$

$$
\bar{\Phi}[S_0, \nabla S_0, \partial_t S_0, v] := \frac{1}{\Phi[S_0, \nabla S_0, \partial_t S_0]} \int_V T_0[S_0, v']T_1[S_0, v, v']dv'.
$$

From $\mathcal{T}_0[S_0](f_0) = 0$, we have

$$
f_0(v, x, t) = \frac{\phi[S_0, \nabla S_0, \partial_t S_0, v]\rho_0(x, t)}{\Phi[S_0, \nabla S_0, \partial_t S_0]},
$$

and therefore it is easy to see $\hat{\Phi}(v) = \bar{\Phi}(v)\rho_0$.

Due to $\mathcal{T}_0[S_0](f_1) = -\mathcal{T}_1[S_0](f_0) - v \cdot \nabla f_0$, we have

$$
\begin{aligned}
f_1(v, x, t) = \frac{1}{\Phi[S_0, \nabla S_0, \partial_t S_0]}(&- v \cdot \nabla f_0(v, x, t) - \tilde{\Phi}[S_0, \nabla S_0, \partial_t S_0, v]f_0(v, x, t) \\
&+ \hat{\Phi}[S_0, \nabla S_0, \partial_t S_0, v]).
\end{aligned}
$$

Computing $J_\epsilon = \int_V v f_1(v, x, t)dv$, we obtain

$$
\begin{aligned}
J_\epsilon = &- \int_V \frac{v^i v^j \partial_{x_j} f_0}{\Phi[S_0, \nabla S_0, \partial_t S_0]}dv - \int_V \frac{v^i \tilde{\Phi}[S_0, \nabla S_0, \partial_t S_0, v]f_0}{\Phi[S_0, \nabla S_0, \partial_t S_0]}dv \\
&+ \int_V \frac{v^i \hat{\Phi}[S_0, \nabla S_0, \partial_t S_0, v]\rho_0}{\Phi[S_0, \nabla S_0, \partial_t S_0]}dv.
\end{aligned}
$$

(5.2)

The first integral in (5.2) becomes

$$
\begin{aligned}
\int_V \frac{v^i v^j \partial_{x_j} f_0}{\Phi[S_0, \nabla S_0, \partial_t S_0]}dv &= \frac{\rho_0}{\Phi[S_0, \nabla S_0, \partial_t S_0]} \int_V \left(v^i v^j \partial_{x_j}\left(\frac{\phi[S_0, \nabla S_0, \partial_t S_0, v]}{\Phi[S_0, \nabla S_0, \partial_t S_0]}\right)\right)dv \\
&\quad + \frac{\partial_{x_j}\rho_0}{\Phi^2[S_0, \nabla S_0, \partial_t S_0]} \int_V v^i v^j \phi[S_0, \nabla S_0, \partial_t S_0, v]dv \\
&= \frac{A_i}{\Phi}\rho_0 + \frac{B_{ij}}{\Phi^2}\partial_{x_j}\rho_0,
\end{aligned}
$$

where

$$(5.3) \qquad A_i = A_i[S_0, \nabla S_0, \partial_t S_0] = \int_V v^i v^j \partial_{x_j} \left(\frac{\phi[S_0, \nabla S_0, \partial_t S_0, v]}{\Phi[S_0, \nabla S_0, \partial_t S_0]} \right) dv,$$

$$(5.4) \qquad B_{ij} = B_{ij}[S_0, \nabla S_0, \partial_t S_0] = \int_V v^i v^j \phi[S_0, \nabla S_0, \partial_t S_0, v] dv.$$

The second integral in (5.2) leads to

$$\int_V \frac{v^i \tilde{\Phi}[S_0, \nabla S_0, \partial_t S_0, v] f_0(v)}{\Phi[S_0, \nabla S_0, \partial_t S_0]} dv = \frac{C_i}{\Phi^2(S_0, \nabla S_0, \partial_t S_0)} \rho_0,$$

where

$$(5.5) \quad C_i = C_i[S_0, \nabla S_0, \partial_t S_0] = \int_V v^i \tilde{\Phi}[S_0, \nabla S_0, \partial_t S_0, v] \phi[S_0, \nabla S_0, \partial_t S_0, v] dv.$$

The last integral in (5.2) becomes

$$\int_V \frac{v^i \hat{\Phi}[S_0, \nabla S_0, \partial_t S_0, v]}{\Phi[S_0, \nabla S_0, \partial_t S_0]} dv = \int_V \frac{v^i \bar{\Phi}[S_0, \nabla S_0, \partial_t S_0, v] \rho_0}{\Phi[S_0, \nabla S_0, \partial_t S_0]} dv = \frac{D_i}{\Phi} \rho_0,$$

where

$$(5.6) \qquad D_i = D_i[S_0, \nabla S_0, \partial_t S_0] = \int_V v^i \bar{\Phi}[S_0, \nabla S_0, \partial_t S_0, v] dv.$$

Summing up, we obtain the macroscopic equation

$$\partial_t \rho_0 = \partial_{x_i} \left(\frac{A_i}{\Phi} \rho_0 + \frac{B_{ij}}{\Phi^2} \partial_{x_j} \rho_0 + \frac{C_i}{\Phi^2} \rho_0 - \frac{D_i}{\Phi} \rho_0 \right), \quad \Phi = \Phi[S_0, \nabla S_0, \partial_t S_0],$$

where A_i, B_{ij}, C_i, and D_i are defined in (5.3)–(5.6).

Remark 5.2. If we drop out the explicit dependence of the last argument v in the functional ϕ in (5.1), then the term $\psi(v \cdot v'/|v||v'|)$ does not influence the resulting macroscopic equation anymore. This is due to the fact that only C_i and D_i depend on ψ (A_i, B_i do not), and $C_i = D_i = 0$ when ϕ is independent of v. This is to be expected from a biological point of view since reorientations without any bias cannot have a macroscopic effect.

In the following we will see how to evaluate A_i, B_{ij}, C_i, and D_i more specifically.

Example 5.3 (rigorous for $\alpha \geq 0$). Let

$$(5.7) \qquad T_\epsilon[S] = \phi(S(x + \epsilon v, t), S(x - \epsilon v', t), \nabla S(x + \epsilon v, t), \nabla S(x - \epsilon v', t)),$$

where S satisfies chemo-attractant equation of elliptic type with $\beta > 0$ in two dimensions. Note that $\phi : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is an even function with respect to the variable ∇S, and increasing and decreasing for the first and second argument, respectively. Also assume the structure condition of Assumptions 2.1 and 3.2, i.e.,

$$|T_\epsilon[S](x, v, v', t)| \leq C(1 + S(x + \epsilon v, t) + S(x - \epsilon v', t) + |\nabla S(x + \epsilon v, t)| + |\nabla S(x - \epsilon v', t)|).$$

Using the asymptotic expansion of the turning kernel, i.e., $T_\epsilon[S] = T_0[S] + \epsilon T_1[S] + O(\epsilon^2)$, we can easily see that $T_0[S] = \phi(S(x, t), S(x, t), \nabla S(x, t), \nabla S(x, t))$, and

$$T_1[S] = (\phi_1(S, S, \nabla S, \nabla S)v - \phi_2(S, S, \nabla S, \nabla S)v') \cdot \nabla S$$

$$+ \sum_{i=1}^{2} (\phi_{2+i}(S, S, \nabla S, \nabla S)v - \phi_{4+i}(S, S, \nabla S, \nabla S)v') \cdot \nabla S_{x_i}.$$

Here ϕ_k, $k = 1, 2, \ldots, 6$, indicates differentiation of ϕ with respect to the kth argument. The symmetric $\phi_\epsilon^A[S]$ and antisymmetric part $\phi_\epsilon^S[S]$ of the turning kernel satisfy

$$\phi_\epsilon^S[S] \geq \gamma\big(1 - \epsilon\Lambda(\|\nabla S\|_{W^{1,\infty}(\mathbb{R}^n)})\big)FF', \quad \int_V \frac{\phi_\epsilon^A[S]^2}{F\phi_\epsilon^S[S]}dv' \leq \epsilon^2\Lambda\left(\|\nabla S\|_{W^{1,\infty}(\mathbb{R}^n)}\right),$$

where $\gamma > 0$ and $\Lambda \in L_{\text{loc}}^\infty$ is a nondecreasing function. By asymptotic expansion of f_ϵ and S_ϵ, the leading order equation becomes $f_0(x, v, t) = \rho_0(x, t)/|V|$. Here f_0 is independent of v. Since the ϵ-order equation is

$$\mathcal{T}_0[S_0](f_1) = -(v \cdot \nabla\rho_0)/|V| - \mathcal{T}_1[S_0](f_0),$$

we have to calculate

$$\mathcal{T}_1[S_0](f_0) = -\rho_0(\phi_1 + \phi_2)\nabla S_0 \cdot v - \sum_{i=1}^2 \rho_0(\phi_{2+i} + \phi_{4+i})\nabla S_{0,x_i} \cdot v.$$

Therefore,

$$\mathcal{T}_0[S_0](f_1) = -\frac{v \cdot \nabla\rho_0}{|V|} + \rho_0(\phi_1 + \phi_2)\nabla S_0 \cdot v + \sum_{i=1}^2 \rho_0(\phi_{2+i} + \phi_{4+i})\nabla S_{0,x_i} \cdot v,$$

due to the solvability condition, and thus we get

$$f_1 = -\frac{v \cdot \nabla\rho_0}{|V|^2\phi} + \frac{\rho_0(\phi_1 + \phi_2)\nabla S_0 \cdot v}{|V|\phi} + \frac{\rho_0(\phi_{2+i} + \phi_{4+i})\nabla S_{0,x_i} \cdot v}{|V|\phi}.$$

Let $\mu = \int_V |v|^2 dv$. Using the above results, we obtain the flux density $J_\epsilon = \int_V v f_1 dv + O(\epsilon)$, where

$$J_\epsilon = -\frac{\mu}{2|V|^2}\frac{\nabla\rho_0}{\phi} + \frac{\mu}{2|V|}\frac{(\phi_1 + \phi_2)\rho_0\nabla S_0}{\phi} + \sum_{i=0}^2 \frac{\mu}{2|V|}\frac{(\phi_{2+i} + \phi_{4+i})\rho_0\nabla S_{0,x_i}}{\phi}.$$

Hence the diffusion limit is

(5.8)
$$\frac{\partial}{\partial t}\rho_0 = \nabla \cdot \left(D\nabla\rho_0 - \chi\rho_0\nabla S_0 - \sum_{i=1}^2 \tilde{\chi}_i\rho_0\nabla S_{0,x_i}\right)$$

with

$$D = \frac{\mu}{2|V|^2\phi}, \quad \chi = \frac{\mu(\phi_1 + \phi_2)}{2|V|\phi}, \quad \tilde{\chi}_i = \frac{\mu(\phi_{2+i} + \phi_{4+i})}{2|V|\phi}, \quad i = 1, 2,$$

coupled to $-\Delta S_0 = \rho_0 - \beta S_0$. It is not known whether solutions for the macroscopic equation (5.8) blow up in finite time or not.

Example 5.4. If we choose an appropriate turning kernel, then the classical Keller–Segel model with constant coefficients can also be obtained. Indeed, if the turning kernel (5.7) is replaced by $T_\epsilon[s] = \phi(S(x,t), S(x+\epsilon v, t), \nabla S(x+\epsilon v, t), \nabla S(x - \epsilon v', t))$, then, by following similar computations to those given above, we have

(5.9)
$$\frac{\partial}{\partial t}\rho_0 = \nabla \cdot \left(\frac{\mu}{2|V|^2\phi}\nabla\rho_0 - \frac{\mu\phi_2}{2|V|\phi}\rho_0\nabla S_0 - \sum_{i=1}^2 \frac{\mu(\phi_{2+i} + \phi_{4+i})}{2|V|\phi}\rho_0\nabla S_{0,x_i}\right).$$

Now let

$$(5.10) \qquad \phi(x_1, x_2, x_3, x_4, x_5, x_6) = \varphi(x_2 - x_1) + \varphi(x_5 - x_3) + \varphi(x_6 - x_4),$$

where $\varphi(x) = C_1\sqrt{1 + x^2} + C_2 x$, $C_1 > C_2 > 0$.

Concerning the gradient terms, this example seems a bit artificial, but it shows how higher order terms might cancel out. Since $\varphi(0) = C_1$, $\varphi'(0) = C_2$, we have $\phi = C_1$, $\phi_2 = C_2$, $\phi_3 = \phi_4 = -C_2$, and $\phi_5 = \phi_6 = C_2$. Therefore (5.9) leads to

$$\frac{\partial}{\partial t}\rho_0 = \nabla \cdot \left(\frac{\mu}{2|V|^2 C_1}\nabla\rho_0 - \frac{\mu C_2}{2|V|C_1}\rho_0\nabla S_0 \right),$$

which is the classical version of the Keller–Segel model. The diffusion coefficient and chemotactic sensitivity, respectively, are $D = \mu/(2|V|^2 C_1)$, $\chi = (\mu C_2)/(2|V|C_1)$, which are both constants in this case.

Example 5.5 (rigorous, $\alpha \geq 0$, $\beta > 0$). The next example considers time variations of the chemical S.

$$(5.11) \qquad T_\epsilon = \sigma S(x + \epsilon v, t) + h(\partial_t S(x, t), \nabla S(x, t), v) + C_2,$$

where $\sigma \geq 0$ is a fixed constant and $h : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $n = 2, 3$, is smooth and bounded, say $-C_1 \leq h \leq C_1$ with $0 < C_1 < C_2$. Note that the turning kernel satisfies the structure condition in Assumption 3.2. Skipping the details of the calculations, *the macroscopic equation reads*

$$\partial_t\rho_0 = \nabla \cdot \left(\frac{1}{\sigma S_0|V| + H[S_0]} \left[\nabla\left(\frac{\mu(\sigma S_0 + C_2)}{\sigma S_0|V| + H[S_0]}\rho_0 \right) + (A_{ij}[S_0]\rho_0)_{x_j} \right] \right.$$

$$(5.12) \qquad \left. - \frac{\sigma\mu}{\sigma S_0|V| + H[S_0]}\rho_0\nabla S_0 \right).$$

This equation is rigorously derived with related turning kernel (5.11) since it satisfies Assumption 4.2.

As a specific example, we consider the case

$$h(\partial_t S, \nabla S, v) = C_1\frac{\gamma\partial_t S + v \cdot \nabla S}{\mathcal{N}(S)}, \quad \mathcal{N}(S) = \sqrt{1 + \gamma^2|\partial_t S|^2 + |\nabla S|^2},$$

where γ is a fixed constant. Then one can easily see

$$H[S_0] = \frac{C_1\gamma\partial_t S_0|V|}{\mathcal{N}(S_0)} + C_2|V|, \qquad A_{ij}[S_0] = \frac{C_1\mu\gamma\partial_t S_0}{(\sigma S_0|B_1| + H[S_0])\mathcal{N}(S_0)}.$$

Therefore, *the macroscopic equation* (5.12) can be explicitly calculated, namely, *for $\gamma = 0$ ($H[S_0] = C_2|V|$ and $A_{ij}[S_0] = 0$),*

$$(5.13) \qquad \partial_t\rho_0 = \nabla \cdot \left(\frac{\mu}{(\sigma S_0 + C_2)|V|^2}\nabla\rho_0 - \frac{\sigma\mu}{(\sigma S_0 + C_2)|V|}\rho_0\nabla S_0 \right).$$

On the other hand, if $\sigma = 0$, then the last term in (5.12) vanishes and (5.12) reads

$$\partial_t\rho_0 = \nabla \cdot \left(\frac{1}{H[S_0]}\nabla\left(\frac{\mu C_2}{H[S_0]}\rho_0 \right) + (A_{ij}[S_0]\rho_0)_{x_j} \right),$$

where $A_{ij}[S_0] = \int_V v^i v^j h(\partial_t S, \nabla S, v)/H[S_0]dv$. In case h is odd with respect to v, then $A_{ij} = 0$ in (5.12).

In the next example we discuss the influence of nonlocal terms in h.

Example 5.6 (formal for $\alpha > 0$, rigorous for $\alpha = 0$). Consider $T_\epsilon = \sigma S(x + \epsilon v, t) + h(\partial_t S(x + \epsilon v, t), v \cdot \nabla S(x + \epsilon v, t)) + C_2$, where $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $n = 2, 3$, is smooth and bounded, say $-C_1 \le h \le C_1$ with $0 < C_1 < C_2$. The structure condition in Assumption 3.2 is satisfied.

Again, skipping the detailed calculations, *the macroscopic equation reads*

$$\partial_t \rho_0 = -\nabla \cdot J_\epsilon = \nabla \cdot \left(\frac{1}{\sigma S_0 |V| + H[S_0]} \left(\int_V v_i v_j \partial_{x_j} f_0 dv + K[S_0] \int_V v_i f_0 dv \right. \right.$$
$$\left. \left. - \rho_0 \int_V v_i T_1[S_0, v] dv \right) \right).$$

Next we consider a specific example of the turning kernel above. Let

$$h = h(\partial_t S(x + \epsilon v, t - \epsilon), v \cdot \nabla S(x + \epsilon v, t)) = \frac{C_1 v \cdot \nabla S(x + \epsilon v, t)}{\sqrt{1 + (v \cdot \nabla S(x + \epsilon v, t))^2}}.$$

Therefore, *the macroscopic equation reads*

$$\partial_t \rho_0 = \nabla \cdot \left(\frac{\mu}{(\sigma S_0 + C)|V|^2} \nabla \rho_0 - \frac{\sigma \mu}{(\sigma S_0 + C)|V|} \rho_0 \nabla S_0 \right.$$
$$\left. + \frac{L[S_0](L[S_0]\Delta S_0 - M[S_0]|\nabla S_0|^2 \Delta S_0)}{(\sigma S_0 + C)^2 |V|^2} \rho_0 \nabla S_0 \right),$$

where

(5.14) $\quad L[S_0] = \frac{1}{n} \int_V \frac{|v|^2}{\sqrt{1 + (v \cdot \nabla S_0)^2}} dv, \qquad M[S_0] = \frac{1}{n^2} \int_V \frac{|v|^4}{(1 + (v \cdot \nabla S_0)^2)^{\frac{3}{2}}} dv.$

The third term in the macroscopic equation is completely due to the nonlocal dependencies of h. Compare (5.13) for the local formulation.

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