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The Bishop-Phelps-Bollobás property for bilinear forms and polynomials

This paper is dedicated to the memory of Joram Lindenstrauss and Robert Phelps

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Abstract. For a σ-finite measure μ and a Banach space Y we study the Bishop-Phelps-Bollobás property (BPBP) for bilinear forms on $L_1(\mu)\times Y$, that is, a (continuous) bilinear form on $L_1(\mu)\times Y$ almost attaining its norm at (f_0, y_0) can be approximated by bilinear forms attaining their norms at unit vectors close to (f_0, y_0). In case that Y is an Asplund space we characterize the Banach spaces Y satisfying this property. We also exhibit some class of bilinear forms for which the BPBP does not hold, though the set of norm attaining bilinear forms in that class is dense.

1. Introduction.

The Bishop-Phelps Theorem states the denseness of the set of norm attaining functionals in the (topological) dual of any Banach space [**16**]. Bollobás proved a quantitative version of this result, known nowadays as the Bishop-Phelps-Bollobás Theorem [**17**], which is very useful to study numerical ranges of operators (see for instance [**18**]). As usual, B_X and S_X denote the closed unit ball and the unit sphere of a Banach space X, respectively; X^* denotes the (topological) dual of X. The Bishop-Phelps-Bollobás Theorem can be stated as follows:

Let X be a Banach space and $0<\varepsilon<1$. Given $x\in B_X$ and $x^*\in S_{X^*}$ with $|1-x^*(x)|<\varepsilon^2/4$, there are elements $y\in S_X$ and $y^*\in S_{X^*}$ such that $y^*(y)=1$, $\|y-x\|<\varepsilon$ and $\|y^*-x^*\|<\varepsilon$.

The study of both results in the vector valued case has attracted the interest of many authors. In his pioneering work [**34**] Lindenstrauss studied versions of the Bishop-Phelps Theorem for operators. He gave the first counterexample of Banach spaces X and Y such that the subset $NA(X,Y)$ of norm attaining operators between X and Y is not dense in $L(X,Y)$, the Banach space of all (bounded and linear) operators from X into Y. He also provided either isomorphic or isometric properties of the Banach spaces

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X and Y to obtain positive results. Bourgain generalized one of these results proving that for each Banach space X with the Radon-Nikodým property, $NA(X,Y)$ is dense in $L(X,Y)$ for every Banach space Y [**19**]. Many other interesting results along the same line for classical Banach spaces have been shown (see [**1**], [**2**], [**9**], [**26**], [**28**], [**29**], [**31**], [**39**], [**40**]). For bilinear forms the study of similar results was initiated by Aron, Finet and Werner [**13**]. In this case there are also interesting positive results and some counterexamples (see [**4**], [**10**], [**13**], [**20**], [**21**], [**30**], [**36**]). The survey [**3**] contains the most relevant achievements in the field until 2006.

For versions of the Bishop-Phelps-Bollobás Theorem the situation is quite different. For instance, let us mention that the Radon-Nikodým property on X does not imply a version of the Bishop-Phelps-Bollobás Theorem for operators from X into any Banach space Y. Even in the case of ℓ_1 this result fails. Actually in [**6**] it is characterized the Banach spaces Y for which the Bishop-Phelps-Bollobás Theorem holds for operators from ℓ_1 into Y.

If μ is a σ-finite measure and m is the Lebesgue measure on the unit interval, it was shown in [**12**] that the Bishop-Phelps-Bollobás Theorem holds for operators from $L_1(\mu)$ into $L_\infty(m)$. Another positive results can be found in [**11**] for operators from an Asplund space X into $C(K)$ (K is a compact Hausdorff space) and in [**32**] for operators from c_0 into a uniformly convex space. There is also a version of the Bishop-Phelps-Bollobás Theorem for operators from a uniformly convex space into any Banach space [**33**] and [**7**].

For the space of bilinear forms the parallel problem was initiated by Choi and Song [**23**]. In this case there are only a few results and the answers are quite different from the operator case. For two Banach spaces X and Y, by using the usual identification of the continuous bilinear forms on $X \times Y$ and the space $L(X,Y^*)$ it holds that a version of the Bishop-Phelps-Bollobás Theorem for bilinear forms on $X \times Y$ implies the parallel result for the space $L(X,Y^*)$. The converse is no longer true even for $X = Y = \ell_1$ in view of [**23**] and [**6**, Theorem 4.1]. However, Dai [**24**] proved that the converse holds if Y is uniformly convex (see also [**7**]). In [**7**] the authors proved that there is a version of the Bishop-Phelps-Bollobás Theorem for bilinear forms on a product of uniformly convex Banach spaces. They also gave a characterization of the Banach spaces Y such that the same results holds for bilinear forms on $\ell_1 \times Y$. As a consequence, they obtained a positive result when the space Y is finite dimensional, uniformly smooth, $C(K)$ or $K(H)$ (the space of compact operators on a Hilbert space H). Also in the case $X = \ell_1$ the mentioned characterization shows the difference between the operator and the bilinear cases.

Our intention now is to list the results proved in this paper. Throughout the paper, X and Y denote Banach spaces over the (same) scalar field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). By $L(^2X \times Y)$ we denote the space of continuous bilinear forms on $X \times Y$.

In order to be more precise, we recall the following definition:

Definition 1.1 ([**23**], [**7**]). The pair (X,Y) has the *Bishop-Phelps-Bollobás property for bilinear forms* (BPBP for bilinear forms), if given $\varepsilon > 0$ there exist $\beta(\varepsilon) > 0$ and $\eta(\varepsilon) > 0$ with $\lim_{\varepsilon\to 0^+} \beta(\varepsilon) = 0$ such that for any $A \in S_{L(^2X\times Y)}$, if $(x_0, y_0) \in S_X \times S_Y$ is such that $|A(x_0, y_0)| > 1 - \eta(\varepsilon)$, then there exist $(u_0, v_0) \in S_X \times S_Y$ and $B \in S_{L(^2X\times Y)}$

satisfying the following conditions:

$$|B(u_0, v_0)| = 1, \ \|u_0 - x_0\| < \beta(\varepsilon), \ \|v_0 - y_0\| < \beta(\varepsilon) \text{ and } \|B - A\| < \varepsilon.$$

The outline of the paper is the following. In Section 2 we give a necessary condition on a Banach space Y in order that $(L_1(\mu), Y)$ has the BPBP for bilinear forms, when $L_1(\mu)$ is infinite dimensional. This condition is called the approximate hyperplane series property for the pair (Y, Y^*) and it was introduced in [**7**]. In case that the measure μ is σ-finite and Y is Asplund, we prove that the mentioned condition is also sufficient, obtaining a complete characterization. We deduce several consequences:

(1) $(L_1(\mu), X)$ has the BPBP for bilinear forms for X finite dimensional and for a σ-finite measure μ.
(2) $(L_1(\mu), c_0)$ has the BPBP for bilinear forms whenever μ is σ-finite.
(3) $(L_1(\mu_1), L_1(\mu_2))$ cannot have the BPBP for bilinear forms when μ_1 and μ_2 are arbitrary measures such that $L_1(\mu_1)$ and $L_1(\mu_2)$ are infinite dimensional.

In Section 3 we prove that the space of (continuous) n-homogeneous polynomials $P(^n X; Y)$ has the BPBP for every Banach space Y, when X is uniformly convex. Finally, in Section 4 we provide classes of Banach spaces X and Y for which the pair (X^*, Y^*) does not have the BPBP for separately w^*-continuous bilinear forms, but the BPBP is satisfied for the corresponding operators, which are w^*-w-continuous operators from X^* into Y. Nevertheless, we show that every separately w^*-continuous bilinear form can be approximated by norm attaining bilinear forms in the same class, i.e. this class satisfies the Bishop-Phelps Theorem for bilinear forms.

2. The Bishop-Phelps-Bollobás Theorem for bilinear forms.

To be precise and to understand related results well, we recall the following definition.

Definition 2.1 ([**6**]). The pair (X, Y) has the *Bishop-Phelps-Bollobás property for operators* (BPBP for operators), if given $\varepsilon > 0$ there exist $\beta(\varepsilon) > 0$ and $\eta(\varepsilon) > 0$ with $\lim_{\varepsilon \to 0^+} \beta(\varepsilon) = 0$ such that for any $T \in S_{L(X,Y)}$, if $x_0 \in S_X$ is such that $\|Tx_0\| > 1 - \eta(\varepsilon)$, then there exist a point $u_0 \in S_X$ and an operator $S \in S_{L(X,Y)}$ satisfying the following conditions:

$$\|Su_0\| = 1, \ \|u_0 - x_0\| < \beta(\varepsilon) \ \text{ and } \ \|S - T\| < \varepsilon.$$

In general, it is clear that (X, Y^*) has the BPBP for operators if (X, Y) has BPBP for bilinear forms. As we already mentioned in the introduction, the pair $(L_1(\mu), L_\infty([0,1]))$ satisfies the BPBP for operators, for every σ-finite measure μ [**12**]. However, the subset of norm attaining bilinear forms on $L_1([0,1]) \times L_1([0,1])$ is not dense in the whole class, i.e. this class does not satisfy the Bishop-Phelps Theorem for bilinear forms [**20**].

At the beginning of the section we will give a necessary isometric condition on a Banach space Y in order that the pair $(L_1(\mu), Y)$ has the BPBP for bilinear forms. Under this condition we will obtain later a characterization by assuming also some extra isomorphic assumption on Y.

Acosta et al [**7**] characterized the Banach spaces X such that the pair (ℓ_1, X) has the BPBP for bilinear forms. They introduced the approximate hyperplane series property (AHSP) for a pair (X, X^*), and showed that (ℓ_1, X) has the BPBP for bilinear forms if and only if the pair (X, X^*) has the AHSP.

DEFINITION 2.2 ([**7**]). For a Banach space X, the pair (X, X^*) has the *approximate hyperplane series property* (AHSP) if for every $\varepsilon > 0$ there exist $0 < \delta$, $\eta < \varepsilon$ such that for every convex series $\sum_n \alpha_n$ and for every sequence $(x_k^*) \subset S_{X^*}$ and $x_0 \in S_X$ with

$$\operatorname{Re} \sum_{n=1}^{\infty} \alpha_n x_n^*(x_0) > 1 - \eta$$

there exist a subset $A \subset \mathbb{N}$, $\{z_k^* : k \in A\} \subset S_{X^*}$ and $z_0 \in S_X$ satisfying

(1) $\sum_{k \in A} \alpha_k > 1 - \delta$,
(2) $\|z_0 - x_0\| < \varepsilon$ and $\|z_k^* - x_k^*\| < \varepsilon$ for all $k \in A$,
(3) $z_k^*(z_0) = 1$ for all $k \in A$.

It is easy to check that we may assume in Definition 2.2 that the sequence (x_k^*) is contained in B_{X^*}.

Next we show that if $(L_1(\mu), Y)$ has the BPBP for bilinear forms, then the pair (Y, Y^*) has the AHSP. The converse is not true in general. Further, we characterize the BPBP for bilinear forms on $(L_1(\mu), Y)$ when Y is an Asplund space.

The following simple result will be useful.

LEMMA 2.3 ([**7**, Lemma 3.5]). *Let z be a complex number with $|z| \leq 1$ and $0 < r < 1$. If $\operatorname{Re} z > r$ then $|z - 1|^2 < 2(1 - r)$.*

THEOREM 2.4. *Let Y be a Banach space and suppose that $L_1(\mu)$ is infinite dimensional. If the pair $(L_1(\mu), Y)$ has the* BPBP *for bilinear forms, then (Y, Y^*) has the* AHSP.

PROOF. Given $0 < \varepsilon < 1$, choose $0 < s < 1$ so that $0 < 2(1 - s) < \varepsilon^2/9$. Let $\eta(\varepsilon)$ and $\beta(\varepsilon)$ be the positive numbers that appear in the definition of the BPBP for bilinear forms. We next choose $\delta > 0$ small enough such that $0 < \delta < \varepsilon/3$, $\beta(\delta)/(1 - s) < \varepsilon/3$ and $\eta(\delta) + \delta + 2\beta(\delta) < \varepsilon^2/18$.

Let $y_0 \in S_Y$, $(y_n^*) \subset S_{Y^*}$ and $\sum \alpha_n$ be a convex series satisfying

$$\operatorname{Re} \sum_{n=1}^{\infty} \alpha_n y_n^*(y_0) > 1 - \eta(\delta).$$

Since $L_1(\mu)$ is infinite dimensional, there is a disjoint sequence $\{E_n\}$ of measurable subsets of Ω such that $0 < \mu(E_n) < \infty$ for all n. Let

$$x_n^*(f) := \int_{E_n} f \, d\mu, \qquad (f \in L_1(\mu)),$$

for each $n \in \mathbb{N}$. Clearly we have that $\sum_{n=1}^{\infty} |x_n^*(f)| = \|f\chi_{\bigcup_{k=1}^{\infty} E_k}\|_1 \leq \|f\|_1$ for every $f \in L_1(\mu)$.

We define the continuous bilinear form $A \in S_{L(^2L_1(\mu)\times Y)}$ by

$$A(f,y) = \sum_{n=1}^{\infty} x_n^*(f) y_n^*(y) \quad (f \in L_1(\mu),\ y \in Y).$$

Clearly $\|A\| = 1$. Let $f_0 = \sum_{n=1}^{\infty} \alpha_n(\chi_{E_n}/\mu(E_n)) \in S_{L_1(\mu)}$. We can see that

$$\operatorname{Re} A(f_0, y_0) > 1 - \eta(\delta).$$

Since the pair $(L_1(\mu), Y)$ has the BPBP for bilinear forms, there exist a bilinear form $B \in S_{L(^2L_1(\mu)\times Y)}$ and $(g, z_0) \in S_{L_1(\mu)} \times S_Y$ such that

$$\|B\| = |B(g,z_0)| = 1,\ \|B - A\| < \delta,\ \|g - f_0\| < \beta(\delta) \text{ and } \|z_0 - y_0\| < \beta(\delta). \tag{2.1}$$

Thus

$$\begin{aligned}
&|B(g,z_0) - A(f_0,y_0)| \\
&\quad \leq |B(g,z_0) - A(g,z_0)| + |A(g,z_0) - A(g,y_0)| + |A(g,y_0) - A(f_0,y_0)| \\
&\quad \leq \|B - A\|\|g\|\|z_0\| + \|A\|\|g\|\|z_0 - y_0\| + \|A\|\|g - f_0\|\|y_0\| \\
&\quad < \delta + 2\beta(\delta).
\end{aligned}$$

Therefore

$$\operatorname{Re} B(g,z_0) \geq \operatorname{Re} A(f_0,z_0) - |B(g,z_0) - A(f_0,y_0)| > 1 - \eta(\delta) - \delta - 2\beta(\delta) > 1 - \frac{\varepsilon^2}{18},$$

and by Lemma 2.3 we have that

$$|1 - B(g,z_0)| < \frac{\varepsilon}{3}. \tag{2.2}$$

Since we know that $\sum_{n=1}^{\infty} |x_n^*(g)| \leq \|g\| \leq 1$, and

$$\begin{aligned}
\beta(\delta) > \|f_0 - g\| &\geq \sum_{n=1}^{\infty} \int_{E_n} |f_0(t) - g(t)|\, d\mu(t) \\
&= \sum_{n=1}^{\infty} \int_{E_n} \left| \frac{\alpha_n}{\mu(E_n)} - g(t) \right| d\mu(t) \geq \sum_{n=1}^{\infty} \left|\alpha_n - \operatorname{Re}\left(x_n^*(g)\right)\right|,
\end{aligned} \tag{2.3}$$

we obtain

$$\sum_{n=1}^{\infty} \operatorname{Re}\left(x_n^*(g)\right) > 1 - \beta(\delta).$$

Defining $C = \{n \in \mathbb{N} : \operatorname{Re}(x_n^*(g)) > s x_n^*(|g|)\}$, we have that

$$
\begin{aligned}
1 - \beta(\delta) &< \sum_{n=1}^{\infty} \operatorname{Re}\big(x_n^*(g)\big) = \sum_{n \in C} \operatorname{Re}\big(x_n^*(g)\big) + \sum_{n \in \mathbb{N}\setminus C} \operatorname{Re}\big(x_n^*(g)\big) \\
&\le \sum_{n \in C} \operatorname{Re}\big(x_n^*(g)\big) + s \sum_{n \in \mathbb{N}\setminus C} x_n^*(|g|) \\
&\le \sum_{n \in C} \operatorname{Re}\big(x_n^*(g)\big) + s\Big(1 - \sum_{n \in C} x_n^*(|g|)\Big) \\
&\le \sum_{n \in C} \operatorname{Re}\big(x_n^*(g)\big) + s\Big(1 - \sum_{n \in C} \operatorname{Re}\big(x_n^*(g)\big)\Big),
\end{aligned}
$$

so

$$
\sum_{n \in C} \operatorname{Re}\big(x_n^*(g)\big) > 1 - \frac{\beta(\delta)}{1-s}.
$$

From this, we can see that $C \neq \emptyset$ and $x_n^*(g) \neq 0$ for all $n \in C$.

Therefore, it follows from (2.3) that

$$
\begin{aligned}
\sum_{n \in C} \alpha_n &\ge \sum_{n \in C} \operatorname{Re}\big(x_n^*(g)\big) - \beta(\delta) \\
&> 1 - \frac{\beta(\delta)}{1-s} - \beta(\delta) = 1 - \gamma(\delta),
\end{aligned}
$$

where we take $\gamma(\delta) := \beta(\delta) + \beta(\delta)/(1-s) < \varepsilon$.

Again by Lemma 2.3 we have for all $n \in C$ that

$$
\left|1 - \frac{x_n^*(g)}{x_n^*(|g|)}\right|^2 < 2(1-s) < \frac{\varepsilon^2}{9}. \tag{2.4}
$$

By (2.1) there is a real number t such that $B(g, z_0) = e^{it}$. For each $n \in \mathbb{N}$ we set $z_n^* = e^{-it} B(g\chi_{E_n}/x_n^*(|g|), \cdot)$ if $x_n^*(|g|) \neq 0$, and $z_n^* = 0$ otherwise. Clearly it is satisfied that $z_n^* \in B_{Y^*}$ for every $n \in \mathbb{N}$. We have

$$
\begin{aligned}
1 = e^{-it} B(g, z_0) &= e^{-it} B\Big(g\chi_{\Omega \setminus (\bigcup_{n=1}^{\infty} E_n)} + \sum_{n=1}^{\infty} g\chi_{E_n}, z_0\Big) \\
&= e^{-it} B\big(g\chi_{\Omega \setminus (\bigcup_{n=1}^{\infty} E_n)}, z_0\big) + \sum_{n=1}^{\infty} x_n^*(|g|) z_n^*(z_0) \\
&\le \big\|g\chi_{\Omega \setminus (\bigcup_{n=1}^{\infty} E_n)}\big\|_1 + \sum_{n=1}^{\infty} x_n^*(|g|) |z_n^*(z_0)|
\end{aligned}
$$

$$\leq \left\| g\chi_{\Omega\setminus(\bigcup_{n=1}^{\infty} E_n)} \right\|_1 + \sum_{n=1}^{\infty} x_n^*(|g|) \leq 1.$$

Thus, $z_n^*(z_0) = 1$ and $\|z_n^*\| = 1$ for every n with $x_n^*(|g|) \neq 0$. Since $A(g\chi_{E_n}/x_n^*(g), \cdot) = y_n^*$ for every $n \in C$, we have

$$\left\| e^{it} z_n^* - \frac{x_n^*(g)}{x_n^*(|g|)} y_n^* \right\| = \left\| B\left(\frac{g\chi_{E_n}}{x_n^*(|g|)}, \cdot \right) - A\left(\frac{g\chi_{E_n}}{x_n^*(|g|)}, \cdot \right) \right\| \leq \|B - A\| < \delta < \frac{\varepsilon}{3}.$$

Therefore, in view of (2.2) and (2.4), for every $n \in C$ it is satisfied that

$$\|z_n^* - y_n^*\| \leq \|z_n^* - e^{it} z_n^*\| + \left\| e^{it} z_n^* - \frac{x_n^*(g)}{x_n^*(|g|)} y_n^* \right\| + \left\| \frac{x_n^*(g)}{x_n^*(|g|)} y_n^* - y_n^* \right\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$

which implies that (Y, Y^*) has the AHSP. □

LEMMA 2.5 ([**7**, Lemma 3.2]). *Let $\{c_n\}$ be a sequence of complex numbers with $|c_n| \leq 1$ for every n, let $\eta > 0$ and $\{\alpha_n\}$ be a sequence of nonnegative real numbers such that $\sum_{n=1}^{\infty} \alpha_n \leq 1$ and assume also that $\operatorname{Re} \sum_{n=1}^{\infty} \alpha_n c_n > 1 - \eta$. Then for every $0 < r < 1$, the set $A := \{i \in \mathbb{N} : \operatorname{Re} c_i > r\}$ satisfies the estimate*

$$\sum_{i \in A} \alpha_i > 1 - \frac{\eta}{1 - r}.$$

If $L_1(\mu)$ is finite dimensional and $N = \dim L_1(\mu)$, by looking carefully the proof of Theorem 2.4 and [**7**, Theorem 3.6], it can be obtained that the pair $(L_1(\mu), Y)$ has the BPBP for bilinear forms if and only if (Y, Y^*) has the AHSP only for finite sums of N elements (instead of any convex series). In case that $L_1(\mu)$ is infinite dimensional we will prove a characterization for Asplund spaces Y of the pairs $(L_1(\mu), Y)$ satisfying the BPBP for bilinear forms under a mild assumption on the measure μ. The argument used to prove this characterization is inspired by the proof of [**22**, Theorem 2.2].

THEOREM 2.6. *Let μ be a σ-finite measure such that $L_1(\mu)$ is infinite dimensional and Y an Asplund space. Then the pair $(L_1(\mu), Y)$ has the* BPBP *for bilinear forms if and only if (Y, Y^*) has the* AHSP.

PROOF. It is enough to prove that if Y is an Asplund space such that (Y, Y^*) satisfies the AHSP, then the pair $(L_1(\mu), Y)$ has the BPBP for bilinear forms. In order to do this, we will denote by $(\Omega, \mathcal{A}, \mu)$ the measure space.

Since (Y, Y^*) does have the AHSP, given $0 < \varepsilon < 1$, there are $0 < \delta, \eta < \varepsilon$ satisfying the conditions in Definition 2.2.

We choose $0 < b < \min\{\eta, 2\delta\}$ and take $a = b\varepsilon/8$. Given $\Phi \in S_{L(^2 L_1(\mu) \times Y)}$, assume that $(f_0, y_0) \in S_{L_1(\mu)} \times S_Y$ satisfies that $|\Phi(f_0, y_0)| > 1 - a$. By using some appropriate linear surjective isometry (say ϕ) on $L_1(\mu)$ and changing the bilinear form Φ by the mapping $(f, y) \mapsto \Phi(\phi^{-1}(f), y)$ we can assume that $f_0(t) \geq 0$ for every $t \in \Omega$. By rotating also the bilinear form Φ, if necessary, we can also assume that $\Phi(f_0, y_0) =$

$|\Phi(f_0, y_0)| > 1 - a$. Let T denote the operator from $L_1(\mu)$ to Y^* associated to the bilinear form Φ. Hence we know that $T \in S_{L(L_1(\mu),Y^*)}$ and it is also satisfied that $T(f_0)(y_0) = \Phi(f_0, y_0) = |\Phi(f_0, y_0)| > 1 - a$.

By the denseness of the simple functions in $L_1(\mu)$, there is a simple positive function $s_0 \in S_{L_1(\mu)}$ satisfying

$$\|s_0 - f_0\|_1 < \varepsilon \quad \text{and also} \quad \operatorname{Re} T(s_0)(y_0) = \operatorname{Re} \Phi(s_0, y_0) > 1 - a. \tag{2.5}$$

So there are a positive integer N, a subset of positive real numbers $\{\alpha_i : i \leq N\}$ and a family of pairwise disjoint subsets $\{A_i : i \leq N\} \subset \mathcal{A}$ satisfying $0 < \mu(A_i) < \infty$ for all i such that

$$s_0 = \sum_{k=1}^{N} \alpha_k \frac{\chi_{A_k}}{\mu(A_k)}, \quad \sum_{k=1}^{N} \alpha_k = 1.$$

Since Y is an Asplund space, Y^* has the Radon-Nikodým property. Also μ is σ-finite, so every operator from $L_1(\mu)$ into Y^* can be represented by a function in $L_\infty(\mu, Y^*)$ (see [**25**, Theorem 5, p. 63, Corollary 3, p. 42]). Hence there is $h \in S_{L_\infty(\mu,Y^*)}$ such that

$$T(f) = \int_\Omega hf \; d\mu, \quad \forall f \in L_1(\mu).$$

Since the range of h is essentially separable, up to an arbitrarily small perturbation, we can also assume that there are a sequence $\{B_n\}$ of pairwise disjoint measurable subsets of Ω such that $\bigcup_{n\in\mathbb{N}} B_n = \Omega$ and functionals $\{y_n^* : n \in \mathbb{N}\} \subset B_{Y^*}$ such that

$$h = \sum_{n=1}^{\infty} \chi_{B_n} y_n^*.$$

If one considers the family of sets $\{A_i \cap B_n : 1 \leq i \leq N, n \in \mathbb{N}, \mu(A_i \cap B_n) > 0\} \cup \{B_n \cap (\Omega \backslash \bigcup_{k\leq N} A_k) : n \in \mathbb{N}\}$, after writing the functions s_0 and h in terms of the above sets and indexing them, we may assume that $A_k = B_k$ for $1 \leq k \leq N$.

We denote by E the set given by

$$E := \{k \leq N : \operatorname{Re} y_k^*(y_0) > 1 - b\}.$$

From the assumption (2.5) we have that

$$\begin{aligned} 1 - a &< \operatorname{Re} T(s_0)(y_0) \\ &= \operatorname{Re} \left(\int_\Omega hs_0 \; d\mu \right)(y_0) \\ &= \operatorname{Re} \left(\sum_{k=1}^{N} \int_{A_k} s_0(t) y_k^*(y_0) \; d\mu \right) \end{aligned}$$

$$= \mathrm{Re} \sum_{k=1}^{N} \alpha_k y_k^*(y_0).$$

By using Lemma 2.5 for $\eta = a$, $r = 1 - b$, we obtain that

$$\|s_0 \chi_{A_E}\|_1 = \sum_{k \in E} \alpha_k > 1 - \frac{a}{b} = 1 - \frac{\varepsilon}{8} > 0 \quad \text{and so} \quad \|s_0 \chi_{\Omega \setminus A_E}\|_1 < \frac{\varepsilon}{8}, \tag{2.6}$$

where $A_E = \bigcup_{k \in E} A_k$.

We denote by $\alpha := \sum_{k \in E} \alpha_k$. We have that

$$\mathrm{Re} \sum_{k \in E} \frac{\alpha_k}{\alpha} y_k^*(y_0) > (1 - b) \sum_{k \in E} \frac{\alpha_k}{\alpha} = 1 - b > 1 - \eta.$$

By using that (Y, Y^*) has the AHSP, there are $C \subset E$, $\{z_k^* : k \in C\} \subset S_{Y^*}$ and $z_0 \in S_Y$ satisfying

$$\sum_{k \in C} \frac{\alpha_k}{\alpha} > 1 - \delta, \quad \|z_0 - y_0\| < \varepsilon, \quad \|z_k^* - y_k^*\| < \varepsilon \quad \text{and} \quad z_k^*(z_0) = 1 \ \ \forall k \in C. \tag{2.7}$$

Consider the function h_1 defined by

$$h_1 = \sum_{k \in C} \chi_{A_k} z_k^* + \sum_{k \in \mathbb{N} \setminus C} \chi_{B_k} y_k^*.$$

Since $h_1 \in B_{L_\infty(\mu, Y^*)}$, this mapping induces an operator $S \in B_{L(L_1(\mu), Y^*)}$ that can be identified with a continuous bilinear form $\Psi \in B_{L(^2 L_1(\mu) \times Y)}$.

In view of (2.7) we also obtain that

$$\begin{aligned} \|\Psi - \Phi\| &= \|S - T\| = \|h_1 - h\|_\infty \\ &= \max_{k \in C} \|z_k^* - y_k^*\| < \varepsilon. \end{aligned}$$

By using again (2.7) and (2.6), the measurable subset $A_C := \bigcup_{k \in C} A_k$ satisfies

$$\|s_0 \chi_{A_C}\|_1 = \sum_{k \in C} \alpha_k > (1 - \delta)\alpha > (1 - \delta)\left(1 - \frac{\varepsilon}{8}\right) > 0. \tag{2.8}$$

We define $g_0 := s_0 \chi_{A_C} / \|s_0 \chi_{A_C}\|_1 \in S_{L_1(\mu)}$. Then we obtain that

$$\begin{aligned} \|g_0 - f_0\|_1 &\le \|g_0 - s_0\|_1 + \|s_0 - f_0\|_1 \\ &< \left\| \frac{s_0 \chi_{A_C}}{\|s_0 \chi_{A_C}\|_1} - s_0 \right\|_1 + \varepsilon \ (\text{by } (2.5)) \end{aligned}$$

$$\begin{aligned}
&\leq \left\| \frac{s_0\chi_{A_C}}{\|s_0\chi_{A_C}\|_1} - s_0\chi_{A_C} \right\|_1 + \|s_0\chi_{\Omega\setminus A_C}\|_1 + \varepsilon \\
&= 1 - \|s_0\chi_{A_C}\|_1 + \|s_0\chi_{\Omega\setminus A_C}\|_1 + \varepsilon \\
&< 2\left(1 - (1-\delta)\left(1 - \frac{\varepsilon}{8}\right)\right) + \varepsilon \text{ (by (2.8))} \\
&= 2\delta - \frac{\delta\varepsilon}{4} + \frac{5\varepsilon}{4} < \frac{13}{4}\varepsilon.
\end{aligned}$$

Let us notice that $\beta(\varepsilon) := (13/4)\varepsilon$ satisfies $\lim_{t\to 0}\beta(t) = 0$.
Finally we have that

$$\begin{aligned}
\Psi(g_0, z_0) = S(g_0)(z_0) &= \left(\int_\Omega h_1 g_0 \, d\mu\right)(z_0) = \left(\int_{A_C} h_1 g_0 \, d\mu\right)(z_0) \\
&= \frac{1}{\|s_0\chi_{A_C}\|_1} \sum_{k\in C} \alpha_k z_k^*(z_0) = \frac{1}{\|s_0\chi_{A_C}\|_1} \sum_{k\in C} \alpha_k = 1.
\end{aligned}$$

We proved that the pair $(L_1(\mu), Y)$ does have the BPBP for bilinear forms with $\beta(\varepsilon) = (13/4)\varepsilon$. □

Acosta et al [**7**] showed that the pair (X, X^*) has the AHSP in the following cases:

(1) X is finite dimensional.
(2) X is uniformly smooth.
(3) X is $C_0(\Omega)$, where Ω is any Hausdorff and locally compact topological space (either real or complex case).
(4) X is $K(H)$, the space of compact operators on a Hilbert space H.

They also showed the following facts:

(1) $(L_1(\mu), L_1(\mu)^*)$ fails to have the AHSP if μ is any measure such that $L_1(\mu)$ is infinite dimensional.
(2) If X is smooth and (X, X^*) has the AHSP, then X is uniformly smooth.

From these we deduce the following result:

Corollary 2.7. *Let μ be a σ-finite measure.*

(1) *Assume that X is a finite dimensional normed space. Then $(L_1(\mu), X)$ has the* BPBP *for bilinear forms. As a consequence, $(X, L_\infty(\mu))$ has the* BPBP *for operators.*
(2) *$(L_1(\mu), c_0)$ has the* BPBP *for bilinear forms, so $(c_0, L_\infty(\mu))$ and $(L_1(\mu), \ell_1)$ have the* BPBP *for operators.*
(3) *Assume that $\dim L_1(\mu) = \infty$ and X is a smooth Banach space. Then $(L_1(\mu), X)$ has the* BPBP *for bilinear forms if and only if X is uniformly smooth.*
(4) *If μ_1 and μ_2 are measures such that $L_1(\mu_1)$ and $L_1(\mu_2)$ are infinite dimensional spaces, the pair $(L_1(\mu_1), L_1(\mu_2))$ fails the* BPBP *for bilinear forms.*

The statement (4) in the previous corollary generalizes the fact that (ℓ_1, ℓ_1) does not have the BPBP for bilinear forms [**23**]. It is worth to remark that Y. S. Choi [**20**] proved that there is no Bishop-Phelps Theorem for bilinear forms on $L_1[0,1] \times L_1[0,1]$. Afterwards Saleh [**38**] showed that the set of norm attaining bilinear forms on $L_1(\mu)$ is dense in the set of all bounded bilinear forms on $L_1(\mu)$ if and only if μ is purely atomic (see also [**37**]).

Related to the assertions (1) and (2) above, it is also worthwhile to notice that Aron, Cascales and Kozhushkina [**11**] studied the BPBP for operators for the case $Y = C_0(L)$ (where L is a locally compact Hausdorff space) and showed that the pair $(X, C_0(L))$ has the BPBP if X is Asplund, and so $(c_0, C_0(L))$ has the BPBP for operators. We also remark that (c_0, Y) has the BPBP for operators for every uniformly convex space Y [**32**].

3. The Bishop-Phelps-Bollobás Theorem for polynomials.

Let $X_1, \dots, X_n$ and Y be either real or complex Banach spaces. As usual, $L(^n X_1 \times \cdots \times X_n; Y)$ will be the subset of (continuous) n-linear mappings from $X_1 \times \cdots \times X_n$ into Y. We will say that the space $L(^n X_1 \times \cdots \times X_n; Y)$ has the *Bishop-Phelps-Bollobás property* (BPBP) if the following condition is satisfied: Given $\varepsilon > 0$ there exist $\beta(\varepsilon) > 0$ and $\eta(\varepsilon) > 0$ with $\lim_{\varepsilon \to 0^+} \beta(\varepsilon) = 0$ such that if $\|A(x_1, \dots, x_n)\| > 1 - \eta(\varepsilon)$ for $A \in S_{L(^n X_1 \times \cdots \times X_n; Y)}$ and $(x_1, \dots, x_n) \in S_{X_1} \times \cdots \times S_{X_n}$, then there exist both an n-linear mapping $B \in S_{L(^n X_1 \times \cdots \times X_n; Y)}$ and $(u_1, \dots, u_n) \in S_{X_1} \times \cdots \times S_{X_n}$ such that

$$\|B(u_1, \dots, u_n)\| = 1, \ \|B - A\| < \varepsilon, \text{ and } \|u_i - x_i\| < \beta(\varepsilon) \text{ for all } 1 \le i \le n.$$

We say that $P(^n X; Y)$ has the *Bishop-Phelps-Bollobás property* (BPBP) when the following condition is satisfied: Given $\varepsilon > 0$ there exist $\beta(\varepsilon) > 0$ and $\eta(\varepsilon) > 0$ with $\lim_{\varepsilon \to 0^+} \beta(\varepsilon) = 0$ such that if $\|Px_0\| > 1 - \eta(\varepsilon)$ for $P \in S_{P(^n X; Y)}$ and $x_0 \in S_X$, then there exist both $Q \in S_{P(^n X; Y)}$ and $u_0 \in S_X$ such that

$$\|Qu_0\| = 1, \ \|u_0 - x_0\| < \beta(\varepsilon) \quad \text{and} \quad \|Q - P\| < \varepsilon.$$

In [**7**], [**33**] it was shown that if X is uniformly convex, the pair (X, Y) has the BPBP for operators for any Banach space Y, and the arguments of the proofs in those papers are different. In fact, in [**7**] it was shown that if $X_1, \dots, X_n$ are uniformly convex, then $L(^n X_1 \times \cdots \times X_n; Y)$ has the Bishop-Phelps-Bollobás property for every Banach space Y. Modifying the argument in [**33**] we will also show this result for the space of homogeneous polynomials.

Theorem 3.1. *Let X be a uniformly convex Banach space. Then $P(^n X; Y)$ has the* BPBP *for every Banach space Y.*

Proof. We will denote by δ the modulus of convexity of the space X. Given $0 < \varepsilon < 1$, $P \in S_{P(^n X; Y)}$ and $x_1 \in S_X$ satisfying that

$$\|Px_1\| > 1 - \frac{\varepsilon}{2^4} \delta\left(\frac{\varepsilon}{2}\right),$$

we can choose $x_1^* \in S_{X^*}$ and $y_1^* \in S_{Y^*}$ satisfying $x_1^*(x_1) = 1$ and

$$|y_1^*(Px_1)| > 1 - \frac{\varepsilon}{2^4}\delta\left(\frac{\varepsilon}{2}\right).$$

We take $P_1 := P$ and define a sequence $(x_k, x_k^*, y_k^*, P_k)_k$ in $S_X \times S_{X^*} \times S_{Y^*} \times S_{P(^nX;Y)}$ inductively. We already have (x_1, x_1^*, y_1^*, P_1). If we assume that (x_k, x_k^*, y_k^*, P_k) was defined and it satisfies that

$$x_k^*(x_k) = 1, \quad |y_k^*(P_k x_k)| > 1 - \frac{\varepsilon}{2^{k+3}}\delta\left(\frac{\varepsilon}{2^k}\right).$$

Set

$$\tilde{P}_{k+1}(x) := P_k(x) + \frac{\varepsilon}{2^{k+2}} x_k^*(x)^n P_k(x_k) \quad (x \in X).$$

Clearly $\tilde{P}_{k+1} \in P(^nX;Y)$ and we also have that

$$\begin{aligned}
\|\tilde{P}_{k+1}\| &\geq |y_k^*(\tilde{P}_{k+1}x_k)| \\
&= |y_k^*(P_k x_k)|\left(1 + \frac{\varepsilon}{2^{k+2}}\right) \\
&> \left(1 - \frac{\varepsilon}{2^{k+3}}\delta\left(\frac{\varepsilon}{2^k}\right)\right)\left(1 + \frac{\varepsilon}{2^{k+2}}\right) \\
&\geq \left(1 - \frac{\varepsilon}{2^{k+3}}\right)\left(1 + \frac{\varepsilon}{2^{k+2}}\right) \\
&> 1.
\end{aligned}$$

In view of the previous estimate and the definition of $\tilde{P}_{k+1}$ we clearly have that

$$1 < \|\tilde{P}_{k+1}\| \leq 1 + \frac{\varepsilon}{2^{k+2}}. \tag{3.1}$$

We can write $P_{k+1} := \tilde{P}_{k+1}/\|\tilde{P}_{k+1}\|$ and choose $x_{k+1} \in S_X$, $x_{k+1}^* \in S_{X^*}$ and $y_{k+1}^* \in S_{Y^*}$ satisfying the following conditions

$$|y_{k+1}^*(\tilde{P}_{k+1}x_{k+1})| > \|\tilde{P}_{k+1}\| - \frac{\varepsilon}{2^{k+4}}\delta\left(\frac{\varepsilon}{2^{k+1}}\right),$$
$$\operatorname{Re} x_k^*(x_{k+1}) = |x_k^*(x_{k+1})| \ \text{ and } x_{k+1}^*(x_{k+1}) = 1.$$

It follows that

$$|y_{k+1}^*(P_{k+1}x_{k+1})| > 1 - \frac{\varepsilon}{2^{k+4}}\delta\left(\frac{\varepsilon}{2^{k+1}}\right).$$

Hence

$$\begin{aligned}\|P_{k+1}-P_k\| &\le \big\|P_{k+1}-\tilde{P}_{k+1}\big\| + \big\|\tilde{P}_{k+1}-P_k\big\|\\ &\le \left\|\frac{\tilde{P}_{k+1}}{\|\tilde{P}_{k+1}\|}-\tilde{P}_{k+1}\right\| + \frac{\varepsilon}{2^{k+2}}\\ &= \big|1-\|\tilde{P}_{k+1}\|\big| + \frac{\varepsilon}{2^{k+2}} < \frac{\varepsilon}{2^{k+1}} \quad \text{(by (3.1))}.\end{aligned}$$

As a consequence, $(P_k)_k$ is a Cauchy sequence and so it converges to some $P_\infty \in S_{P({}^nX;Y)}$ that satisfies $\|P_\infty - P\| < \varepsilon$.

Now we will show that (x_k) is a Cauchy sequence. In order to do this let us notice that

$$\begin{aligned}\big\|\tilde{P}_{k+1}\big\| - \frac{\varepsilon}{2^{k+4}}\delta\left(\frac{\varepsilon}{2^{k+1}}\right) &< \big|y^*_{k+1}\big(\tilde{P}_{k+1}x_{k+1}\big)\big|\\ &= \left|y^*_{k+1}(P_kx_{k+1}) + \frac{\varepsilon}{2^{k+2}}x^*_k(x_{k+1})^n y^*_{k+1}(P_kx_k)\right|\\ &\le 1 + \frac{\varepsilon}{2^{k+2}}\big|x^*_k(x_{k+1})^n\big|\\ &= 1 + \frac{\varepsilon}{2^{k+2}}\big(\operatorname{Re} x^*_k(x_{k+1})\big)^n.\end{aligned}$$

On the other hand, we have the following lower estimate

$$\begin{aligned}\big\|\tilde{P}_{k+1}\big\| &\ge \big|y^*_k\big(\tilde{P}_{k+1}x_k\big)\big|\\ &= \left|y^*_k(P_kx_k) + \frac{\varepsilon}{2^{k+2}}x^*_k(x_k)^n \cdot y^*_k(P_kx_k)\right|\\ &= \left(1+\frac{\varepsilon}{2^{k+2}}\right)\big|y^*_k(P_kx_k)\big|\\ &> \left(1+\frac{\varepsilon}{2^{k+2}}\right)\left(1-\frac{\varepsilon}{2^{k+3}}\delta\left(\frac{\varepsilon}{2^k}\right)\right).\end{aligned}$$

From the previous upper and lower estimates it follows that

$$1-\frac{\varepsilon}{2^{k+3}}\delta\left(\frac{\varepsilon}{2^k}\right)+\frac{\varepsilon}{2^{k+2}}-\frac{\varepsilon^2}{2^{2k+5}}\delta\left(\frac{\varepsilon}{2^k}\right) < 1+\frac{\varepsilon}{2^{k+2}}\big(\operatorname{Re} x^*_k(x_{k+1})\big)^n + \frac{\varepsilon}{2^{k+4}}\delta\left(\frac{\varepsilon}{2^{k+1}}\right),$$

that is,

$$\begin{aligned}1 \ge \big(\operatorname{Re} x^*_k(x_{k+1})\big)^n &> 1-\frac{1}{2}\delta\left(\frac{\varepsilon}{2^k}\right)-\frac{1}{2^2}\delta\left(\frac{\varepsilon}{2^{k+1}}\right)-\frac{1}{2^{k+3}}\delta\left(\frac{\varepsilon}{2^k}\right)\\ &> 1-\delta\left(\frac{\varepsilon}{2^k}\right).\end{aligned}$$

Since $\operatorname{Re} x_k^*(x_{k+1})$ is a nonnegative real number, we obtain that

$$\begin{aligned}\left\|\frac{x_k + x_{k+1}}{2}\right\| &\geq \operatorname{Re}\left(\frac{x_k^*(x_k + x_{k+1})}{2}\right)\\ &\geq 1 - \frac{1}{2}\delta\left(\frac{\varepsilon}{2^k}\right) > 1 - \delta\left(\frac{\varepsilon}{2^k}\right).\end{aligned}$$

This implies that $\|x_{k+1} - x_k\| < \varepsilon/2^k$ and so $(x_k)_k$ is a Cauchy sequence in S_X, that is, converges to some element $x_\infty \in S_X$ that satisfies $\|x_\infty - x_1\| < \varepsilon$.

Since $\lim_k \|P_k x_k\| = 1$, by taking into account that both $(P_k)_k$ and $(x_k)_k$ converge in norm, it follows that $\|P_\infty x_\infty\| = 1$. □

Let us notice that a similar argument also gives the analogous result for n-linear mappings defined on a product of uniformly convex spaces (see also [**7**, Theorem 2.2]). However, we do not know if the analogous result holds for symmetric n-linear forms on a uniformly convex space.

Now we recall an isometric condition, called *property β* that was introduced by Lindenstrauss [**34**].

Definition 3.2. A Banach space Y is said to have *property β* (of Lindenstrauss) if there are two sets $\{y_i : i \in I\} \subset S_Y$, $\{y_i^* : i \in I\} \subset S_{Y^*}$ and $0 \leq \rho < 1$ such that the following conditions hold:

(1) $y_i^*(y_i) = 1$, $\forall i \in I$.
(2) $|y_i^*(y_j)| \leq \rho < 1$ if $i, j \in I, i \neq j$.
(3) $\|y\| = \sup_{i \in I}\{|y_i^*(y)|\}$, for all $y \in Y$.

The spaces c_0 and ℓ_∞ satisfy the above property. In both cases $\rho = 0$.

Given a family $\mathcal{F}$ of mappings defined on a Banach space and bounded on its unit ball, by $\mathrm{NA}(\mathcal{F})$ we will denote the subset of norm attaining elements of $\mathcal{F}$. In case that Y has property β, the following results are shown in [**21**]:

(1) If $\mathrm{NA}(L(^nX))$ is dense in $L(^nX)$, then $\mathrm{NA}(L(^nX;Y))$ is dense in $L(^nX;Y)$.
(2) If $\mathrm{NA}(L_s(^nX))$ is dense in $L_s(^nX)$, then $\mathrm{NA}(L_s(^nX;Y))$ is dense in $L_s(^nX;Y)$, where $L_s(^nX;Y)$ stands for the symmetric n-linear mappings from X into Y.
(3) If $\mathrm{NA}(P(^nX))$ is dense in $P(^nX)$, then $\mathrm{NA}(P(^nX;Y))$ is dense in $P(^nX;Y)$.

In what follows we can show similar results for the Bishop-Phelps-Bollobás property.

Proposition 3.3. *Suppose that the Banach space Y has property β.*

(1) *If $L(^nX_1 \times \cdots \times X_n)$ has the* BPBP*, then $L(^nX_1 \times \cdots \times X_n; Y)$ has the* BPBP.
(2) *If $P(^nX)$ has the* BPBP*, then $P(^nX;Y)$ has the* BPBP.

Proof. The proofs of parts (1) and (2) are almost the same, hence we show only the proof of (1). Assume that $L(^nX_1 \times \cdots \times X_n)$ has the BPBP. Given $0 < \varepsilon < 1/2$, let $\eta(\varepsilon)$ and $\beta(\varepsilon)$ be the functions in the definition of the BPBP. Take $0 < \varepsilon' < \varepsilon(1-\rho)$ and put $\eta'(\varepsilon) = \min\{\eta(\varepsilon'), ((1-\rho)\varepsilon - \rho\varepsilon')/(1+\varepsilon)\}$, where ρ is the positive real number satisfying Definition 3.2.

Let $A \in S_{L(^n X_1 \times \cdots \times X_n; Y)}$ and $(x_1^0, \ldots, x_n^0) \in \prod_{j=1}^n S_{X_j}$ satisfying that

$$\big\|A(x_1^0, \ldots, x_n^0)\big\| > 1 - \eta'(\varepsilon).$$

Choose one of the functionals $y_{i_0}^*$ appearing in Definition 3.2 so that

$$\big|y_{i_0}^* A(x_1^0, \ldots, x_n^0)\big| > 1 - \eta'(\varepsilon) \geq 1 - \eta(\varepsilon').$$

By the assumption, there exist $\varphi \in L(^n X_1 \times \cdots \times X_n)$ and $(\tilde{x}_1, \ldots, \tilde{x}_n)$ satisfying

$$1 - \eta'(\varepsilon) \leq \|\varphi\| \leq 1, \quad \|y_{i_0}^* A - \varphi\| < \varepsilon',$$
$$\sup_j \big\|\tilde{x}_j - x_j^0\big\| < \beta(\varepsilon') \quad \text{and} \quad |\varphi(\tilde{x}_1, \ldots, \tilde{x}_n)| = \|\varphi\|.$$

Define

$$B(x_1, \ldots, x_n) = A(x_1, \ldots, x_n) + \big[((1+\varepsilon)\varphi - y_{i_0}^* A)(x_1, \ldots, x_n)\big] y_{i_0},$$

for all $(x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n$. It is clear that $B \in L(^n X_1 \times \cdots \times X_n; Y)$. We can see that

$$\|B - A\| \leq \varepsilon \|\varphi\| + \|y_{i_0}^* A - \varphi\| < \varepsilon + \varepsilon' < 1.$$

Hence $B \neq 0$ and

$$\Big\|\frac{B}{\|B\|} - A\Big\| < 4\varepsilon.$$

Moreover, we obtain for every $i \neq i_0$

$$\begin{aligned} \|y_i^* B\| &\leq \|A\| + |y_i^*(y_{i_0})| \big(\varepsilon \|\varphi\| + \|y_{i_0}^* A - \varphi\|\big) \\ &< 1 + \rho(\varepsilon + \varepsilon'), \end{aligned}$$

and

$$\begin{aligned} \|y_{i_0}^* B\| = (1+\varepsilon)\|\varphi\| &\geq (1+\varepsilon)(1 - \eta'(\varepsilon)) \\ &\geq 1 + \rho(\varepsilon + \varepsilon'). \end{aligned}$$

The following inequality implies that B attains its norm at $(\tilde{x}_1, \ldots, \tilde{x}_n)$

$$\begin{aligned} \|B\| = \|y_{i_0}^* B\| = (1+\varepsilon)\|\varphi\| &= \big|(1+\varepsilon)\varphi(\tilde{x}_1, \ldots, \tilde{x}_n)\big| \\ &= \big|y_{i_0}^* B(\tilde{x}_1, \ldots, \tilde{x}_n)\big| \leq \|B(\tilde{x}_1, \ldots, \tilde{x}_n)\| \\ &\leq \|B\|. \end{aligned}$$

Hence, $\eta'(\varepsilon/4)$ and $\beta(\varepsilon'/4)$ are the functions that we have found. $\square$

Remark 3.4. An analogous argument used in the above proof also works for the symmetric case. However we do not know of any infinite dimensional Banach space X such that the Bishop-Phelps-Bollobás property is satisfied for the symmetric n-linear forms on X.

4. Results for bilinear forms that are separately w^*-continuous on the product of dual Banach spaces.

In this section we will provide classes of Banach spaces X and Y for which the pair (X^*, Y^*) does not have the BPBP for separately w^*-continuous bilinear forms, though the BPBP is satisfied for the corresponding operators, which are w^*-w-continuous operators from X^* into Y. Nevertheless, we will show that every separately w^*-continuous bilinear form can be approximated by norm attaining bilinear forms in the same class.

For $x^* \in S_{X^*}$ and $0 < \varepsilon < 1$, the set of the form

$$S(A, x^*, \varepsilon) = \left\{x \in B_X : \operatorname{Re} x^*(x) > \sup_{x \in A} \operatorname{Re} x^*(x) - \varepsilon\right\}$$

is called a *slice* of $A \subset X$.

The following technical result appears implicitly in [**15**, Proposition 3.1, Claim]. We isolate the statement and include its proof.

Lemma 4.1. *Let A be a subset of X^* such that $B_{X^*} = \overline{\operatorname{co}}^{w^*}(A)$, and let $0 < \varepsilon < 1$ and $x \in S_X$. For every $x^* \in S(B_{X^*}, x, \varepsilon^2)$, there exists $z^* \in \operatorname{co}(S(A, x, \varepsilon))$ such that $\|z^* - x^*\| \leq 3\varepsilon$.*

Proof. Fix $0 < \varepsilon < 1$ and $x \in S_X$. First we will show that for every $x^* \in S(\operatorname{co}(A), x, \varepsilon^2)$, there is $y^* \in \operatorname{co}(S(A, x, \varepsilon))$ such that $\|y^* - x^*\| \leq 2\varepsilon$.

If $x^* \in S(\operatorname{co}(A), x, \varepsilon^2)$, there exist $n \in \mathbb{N}$, $\{a_i^* : 1 \leq i \leq n\} \subset A$ and $\{t_i : 1 \leq i \leq n\} \subset [0, 1]$ with $\sum_{i=1}^n t_i = 1$ such that $x^* = \sum_{i=1}^n t_i a_i^*$. Now let

$$B := \{i \leq n : \operatorname{Re} a_i^*(x) > 1 - \varepsilon\}, \quad C := \{1, \dots, n\} \backslash B.$$

By Lemma 2.5 we have that

$$r := \sum_{i \in B} t_i > 1 - \frac{\varepsilon^2}{1 - (1 - \varepsilon)} = 1 - \varepsilon.$$

If we define $y^* := (1/r) \sum_{i \in B} t_i a_i^*$, then $y^* \in \operatorname{co}(S(A, x, \varepsilon))$ and

$$\begin{aligned}\|y^* - x^*\| &= \left\|\frac{1}{r} \sum_{i \in B} t_i a_i^* - \sum_{i=1}^n t_i a_i^*\right\| \\ &\leq \left(\frac{1}{r} - 1\right) \left\|\sum_{i \in B} t_i a_i^*\right\| + \left\|\sum_{i \in C} t_i a_i^*\right\|\end{aligned}$$

$$\leq 2(1-r) \leq 2\varepsilon.$$

Now let $x^* \in S(B_{X^*}, x, \varepsilon^2)$. Since $\operatorname{co}(A)$ is w^*-dense in B_{X^*}, there is a net $\{x^*_\lambda\}_{\lambda\in\Lambda}$ in $S(\operatorname{co}(A), x, \varepsilon^2)$ such that $\{x^*_\lambda\}_{\lambda\in\Lambda} \overset{w^*}{\to} x^*$. We know that for each λ, there is $y^*_\lambda \in \operatorname{co}(S(A,x,\varepsilon))$ such that $\|y^*_\lambda - x^*_\lambda\| \leq 2\varepsilon$. Now, let $y^* \in B_{X^*}$ be a w^*-cluster point of the net $\{y^*_\lambda\}_{\lambda\in\Lambda}$. Clearly $y^* \in \overline{\operatorname{co}}^{w^*}(S(A,x,\varepsilon))$. Since the dual norm is w^*-lower semi-continuous, we conclude that $\|y^* - x^*\| \leq \liminf_\lambda \|y^*_\lambda - x^*_\lambda\| \leq 2\varepsilon$. By using again the w^*-lower semicontinuity of the dual norm, there is $z^* \in \operatorname{co}(S(A,x,\varepsilon))$ such that $\|z^* - x^*\| \leq 3\varepsilon$. □

If X and Y are Banach spaces, we will denote by $L_{w^*}({}^2X^* \times Y^*)$ the Banach space of all separately w^*-continuous bilinear forms on the product $X^* \times Y^*$.

Proposition 4.2. *Let X and Y be Banach spaces. Then the set of norm attaining bilinear forms in $L_{w^*}({}^2X^* \times Y^*)$ is dense.*

Proof. Let $A \in L_{w^*}({}^2X^* \times Y^*)$ and $0 < \varepsilon < 1/3$ be given. Since the subset of norm attaining bilinear forms is stable under product by scalars, we may assume that $\|A\| = 1$. We choose a decreasing sequence $\{\varepsilon_k\}$ of positive numbers satisfying the following conditions:

$$2\sum_{i=1}^{\infty}\varepsilon_i < \varepsilon, \quad 2\sum_{i=k+1}^{\infty}\varepsilon_i < \varepsilon_k^2, \quad \text{and } \varepsilon_k < 1/10k,\ \forall k \in \mathbb{N}.$$

We next construct inductively a sequence $(A_k)_{k=1}^\infty \subset L_{w^*}({}^2X^* \times Y^*)$ as follows. Take $A_1 = A$; if we assume that we have already defined $A_k \in L_{w^*}({}^2X^* \times Y^*)$, we choose functionals $x^*_k \in S_{X^*}$ and $y^*_k \in S_{Y^*}$, satisfying

$$A_1 = A, \quad \operatorname{Re} A_k(x^*_k, y^*_k) = |A_k(x^*_k, y^*_k)| \geq \|A_k\| - \varepsilon_k^2$$

and we define

$$A_{k+1}(x^*, y^*) := A_k(x^*, y^*) + \varepsilon_k A_k(x^*, y^*_k) A_k(x^*_k, y^*), \quad (x^* \in X^*, y^* \in Y^*).$$

Clearly we construct by this procedure a sequence (A_k) in $L_{w^*}({}^2X^* \times Y^*)$. In [**14**, Theorem 2.2(2)] it is proved that the sequence (A_j) converges in norm to a bilinear form B satisfying $\|B - A\| \leq \varepsilon$ and also

$$\left|B(x^*_k, y^*_j)\right| \geq \|B\| - \frac{1}{j}, \quad \forall k > j.$$

It is also clear in this case that B is w^*-separately continuous, since it is the uniform limit on bounded sets of $X^* \times Y^*$ of a sequence in $L_{w^*}({}^2X^* \times Y^*)$. By Banach-Alaoglu's theorem there is a w^*-cluster point $x^*_0 \in B_{X^*}$ of the sequence (x^*_k). Applying that B is w^*-separately continuous, we obtain

$$\left|B(x_0^*, y_j^*)\right| \geq \|B\| - \frac{1}{j}, \quad \forall j \in \mathbb{N}.$$

An analogous argument shows that there exists $y_0^* \in B_{Y^*}$ such that

$$|B(x_0^*, y_0^*)| = \|B\|,$$

and so B attains its norm. □

If X and Y are Banach spaces, we denote by $L_{w^*\text{-}w}(X^*, Y)$ the Banach space of all w^*-w-continuous linear operators from X^* into Y, which can be identified with $L_{w^*}({}^2X^* \times Y^*)$. Actually, given $\varphi \in L_{w^*}({}^2X^* \times Y^*)$, we define an operator $T : X^* \to Y^{**}$ by $T(x^*)(y^*) := \varphi(x^*, y^*)$. Since φ is separately w^*-continuous, for every $x^* \in X^*$, $T(x^*)$ is a w^*-continuous functional on Y^*, hence $T(x^*) \in Y$. Again by using that φ is separately w^*-continuous on the first variable, it follows that T is w^*-w-continuous. Conversely, given $T \in L_{w^*\text{-}w}(X^*, Y)$, we define $\varphi \in L_{w^*}({}^2X^* \times Y^*)$ by $\varphi(x^*, y^*) := y^*(Tx^*)$. Since T is w^*-w-continuous, φ is separately w^*-continuous.

Since an operator attains its norm whenever the corresponding bilinear form does, we obtain the next Bishop-Phelps result for operators, that was already proved by Manuel Ruiz Galán in a similar way.

Corollary 4.3. *Let X and Y be Banach spaces. Then the set of norm attaining operators in $L_{w^*\text{-}w}(X^*, Y)$ is dense. In fact, we have that an operator in $L_{w^*\text{-}w}(X^*, Y)$ attains its norm if and only if the associate separately w^*-continuous bilinear form attains its norm.*

The next result improves the Bishop-Phelps result stated above for $L_{w^*\text{-}w}(X^*, Y)$ under some additional assumption on Y.

Proposition 4.4. *Let X and Y be Banach spaces, and assume that Y has property β. If $T \in S_{L_{w^*\text{-}w}(X^*,Y)}$, $\varepsilon > 0$, and $x_0^* \in S_{X^*}$ satisfy $\|T(x_0^*)\| > 1 - \varepsilon^2/4$, then for each real number η such that $\eta > (\rho/(1-\rho))(\varepsilon + (\varepsilon^2/4))$, there are $S \in L_{w^*\text{-}w}(X^*, Y)$, $z_0^* \in S_{X^*}$ such that*

$$\|S(z_0^*)\| = \|S\|, \quad \|z_0^* - x_0^*\| < \varepsilon, \quad \textit{and} \quad \|S - T\| < \eta + \varepsilon + \frac{\varepsilon^2}{4}.$$

Proof. Since Y has property β, there is $\alpha_0 \in \Lambda$ such that $|y_{\alpha_0}^*(T(x_0^*))| > 1 - (\varepsilon^2/4)$. Since T is w^*-w-continuous, we have that $T^*(y_{\alpha_0}^*) \in X$. By the Bishop-Phelps-Bollobás Theorem, there exist $z_0^* \in S_{X^*}$ and $z_0 \in S_X$ such that $|z_0^*(z_0)| = 1$, $\|z_0^* - x_0^*\| < \varepsilon$ and $\|z_0 - (T^*(y_{\alpha_0}^*)/\|T^*(y_{\alpha_0}^*)\|)\| < \varepsilon$. For a real number η satisfying $\eta > (\rho/(1-\rho))(\varepsilon + (\varepsilon^2/4))$, we define an operator $S \in L(X^*, Y)$ by

$$S(x^*) = T(x^*) + \big((1+\eta)x^*(z_0) - x^*(T^*(y_{\alpha_0}^*))\big)y_{\alpha_0}, \quad (x^* \in X^*),$$

that is clearly w^*-w-continuous. The proof can be finished if one follows the same steps as in the proof of [**6**, Theorem 2.2]. □

However, we will show that in general the BPBP is not satisfied for the space of w^*-separately continuous bilinear forms.

THEOREM 4.5. *Let X and Y be infinite dimensional Banach spaces having property β with $\rho = 0$. Then the pair (X^*, Y^*) does not satisfy the Bishop-Phelps-Bollobás property for separately w^*-continuous bilinear forms.*

PROOF. We will argue by contradiction, so assume that the pair (X^*, Y^*) satisfies the Bishop-Phelps-Bollobás property for separately w^*-continuous bilinear forms. Given $\varepsilon > 0$, there are $\eta(\varepsilon) > 0$ and $\beta(\varepsilon) > 0$ with $\lim_{\varepsilon\to 0}\beta(\varepsilon) = 0$ such that for all $A \in S_{L_{w^*}(^2X^*\times Y^*)}$ and $x_0^* \in S_{X^*}$, $y_0^* \in S_{Y^*}$ with $|A(x_0^*, y_0^*)| > 1 - \eta(\varepsilon)$, there exist elements $u_0^* \in S_{X^*}$, $v_0^* \in S_{Y^*}$, and $B \in S_{L_{w^*}(^2X^*\times Y^*)}$ satisfying the following conditions:

$$|B(u_0^*, v_0^*)| = 1, \quad \|u_0^* - x_0^*\| < \beta(\varepsilon), \quad \|v_0^* - y_0^*\| < \beta(\varepsilon) \quad \text{and} \quad \|B - A\| < \varepsilon.$$

We fix $\varepsilon \in (0, 1)$ such that $\beta(\varepsilon) < 1/2$, and $n \in \mathbb{N}$ such that $1/2n^2 < \eta(\varepsilon)$. Since the Banach spaces X and Y have property β with $\rho = 0$, there exist sets $\{x_\alpha : \alpha \in \Lambda\} \subset S_X$ and $\{x_\alpha^* : \alpha \in \Lambda\} \subset S_{X^*}$, and sets $\{y_\gamma : \gamma \in \Gamma\} \subset S_Y$ and $\{y_\gamma^* : \gamma \in \Gamma\} \subset S_{Y^*}$ satisfying the conditions of Definition 3.2. Since X and Y are infinite dimensional spaces, Λ and Γ are infinite sets. So we can choose $2n^2$ elements of the subsets $\{x_\alpha : \alpha \in \Lambda\}$ and $\{y_\gamma : \gamma \in \Gamma\}$, that we will simply denote by $\{x_i : i = 1, \ldots, 2n^2\}$ and $\{y_j : j = 1, \ldots, 2n^2\}$, respectively. We will also denote by $\{x_i^* : 1 \le i \le 2n^2\}$ and $\{y_j^* : 1 \le j \le 2n^2\}$ the functionals appearing in Definition 3.2 associated to the previous subsets of elements in X and Y, respectively.

Now we define a bilinear form $A \in L(^2X^* \times Y^*)$ by

$$A(x^*, y^*) := \sum_{i=1}^{2n^2} x^*(x_i)\Bigg(\sum_{j=1, j\neq i}^{2n^2} y^*(y_j)\Bigg) \quad ((x^*, y^*) \in X^* \times Y^*),$$

that clearly belongs to $L_{w^*}(^2X^* \times Y^*)$. Since $\{x_\alpha^* : \alpha \in \Lambda\}$ and $\{y_\gamma^* : \gamma \in \Gamma\}$ are norming sets for X and Y, respectively, and A is separately w^*-continuous, we have that

$$\|A\| = \sup\big\{|A(x_\alpha^*, y_\gamma^*)| : \alpha \in \Lambda, \gamma \in \Gamma\big\}.$$

By assumption X and Y have property β with $\rho = 0$ and so $\|A\| = 1$.

We consider the elements $x_0^* := \sum_{i=1}^{2n^2}(1/2n^2)x_i^* \in S_{X^*}$ and $y_0^* := \sum_{i=1}^{2n^2}(1/2n^2)y_i^* \in S_{Y^*}$. It is immediate to check that $A(x_0^*, y_0^*) = 1 - (1/2n^2)$ and so $|A(x_0^*, y_0^*)| > 1 - \eta(\varepsilon)$. By using the assumption, there exist elements $u_0^* \in S_{X^*}$, $v_0^* \in S_{Y^*}$, and a bilinear form $B \in S_{L_{w^*}(^2X^*\times Y^*)}$ such that

$$|B(u_0^*, v_0^*)| = 1, \quad \|u_0^* - x_0^*\| < \beta(\varepsilon), \quad \|v_0^* - y_0^*\| < \beta(\varepsilon) \quad \text{and} \quad \|B - A\| < \varepsilon.$$

We choose a scalar λ_0 with modulus one such that $B(u_0^*, \lambda_0 v_0^*) = 1$. Since B is separately w^*-continuous and $\|B\| = 1$, there exists $x_0 \in B_X$ such that

$$x^*(x_0) = B(x^*, \lambda_0 v_0^*), \quad \forall x^* \in X^*.$$

Let $0 < \delta < 1 - \varepsilon$ and $C := \{\lambda x_\alpha^* : \lambda \in \mathbb{K}, |\lambda| \leq 1, \alpha \in \Lambda\}$. Since $\operatorname{Re} B(x^*, \lambda_0 v_0^*) > 1 - \delta$ for every x^* in $S(C, x_0, \delta)$ and $\|B - A\| < \varepsilon$, we obtain $A(x^*, \lambda_0 v_0^*) \neq 0$ for each $x^* \in S(C, x_0, \delta)$. This implies that

$$S(C, x_0, \delta) \subseteq \{\lambda x_i^* : \lambda \in \mathbb{K}, |\lambda| \leq 1, 1 \leq i \leq 2n^2\}. \tag{4.1}$$

Since $B(u_0^*, \lambda_0 v_0^*) = 1$, it is immediate that $u_0^* \in S(B_{X^*}, x_0, \delta^2)$. By assumption X has property β and in view of condition (3) in Definition 3.2 we know that the subset C satisfies $B_{X^*} = \overline{\text{co}}^{w^*}(C)$. By Lemma 4.1 there exists $u^* \in \text{co}(S(C, x_0, \delta))$ such that $\|u^* - u_0^*\| \leq 3\delta$.

Since δ can be taken arbitrarily small, in view of (4.1) we conclude that $u_0^* \in \text{aco}\{x_i^* : 1 \leq i \leq 2n^2\}$, where $\text{aco}(E)$ means the absolutely convex hull of a subset E of a linear space. Arguing similarly, we have that $v_0^* \in \text{aco}\{y_i^* : 1 \leq i \leq 2n^2\}$. Hence there exist scalars t_i and s_i $(1 \leq i \leq 2n^2)$ with $\sum_{i=1}^{2n^2} |t_i| \leq 1, \sum_{i=1}^{2n^2} |s_i| \leq 1$ and such that $u_0^* = \sum_{i=1}^{2n^2} t_i x_i^*$ and $v_0^* = \sum_{i=1}^{2n^2} s_i y_i^*$. Since

$$\begin{aligned} 1 = |B(u_0^*, v_0^*)| &= \left| \sum_{i,j=1}^{2n^2} t_i s_j B(x_i^*, y_j^*) \right| \leq \sum_{i,j=1}^{2n^2} |t_i s_j| \, |B(x_i^*, y_j^*)| \\ &\leq \left(\sum_{i=1}^{2n^2} |t_i| \right) \left(\sum_{j=1}^{2n^2} |s_j| \right) = 1, \end{aligned}$$

we have that $|B(x_i^*, y_j^*)| = 1$ for all $i \in G(u_0^*) := \{i \leq 2n^2 : t_i \neq 0\}$ and for all $j \in G(v_0^*) := \{j \leq 2n^2 : s_j \neq 0\}$. Since $A(x_i^*, y_i^*) = 0$ for all $i = 1, \ldots, 2n^2$ and $\|B - A\| < \varepsilon$, it follows that $G(u_0^*) \cap G(v_0^*)$ is empty. We consider the elements given by

$$x_0 := \sum_{\substack{i=1 \\ i \notin G(u_0^*)}}^{2n^2} x_i \quad \text{and} \quad y_0 := \sum_{\substack{j=1 \\ j \notin G(v_0^*)}}^{2n^2} y_j.$$

Clearly $x_0 \in B_X$ and $y_0 \in B_Y$. We obtain that

$$\sum_{\substack{i=1 \\ i \notin G(u_0^*)}}^{2n^2} \frac{1}{2n^2} = |x_0^*(x_0)| = |(x_0^* - u_0^*)(x_0)| \leq \|x_0^* - u_0^*\| < \beta(\varepsilon) < \frac{1}{2}$$

and

$$\sum_{\substack{i=1 \\ i \notin G(v_0^*)}}^{2n^2} \frac{1}{2n^2} = |y_0^*(y_0)| = |(y_0^* - v_0^*)(y_0)| \leq \|y_0^* - v_0^*\| < \beta(\varepsilon) < \frac{1}{2}$$

and hence the sets $G(u_0^*)$ and $G(v_0^*)$ contain more than n^2 elements. This is impossible, since $G(u_0^*) \cap G(v_0^*) = \emptyset$ and both subsets are contained in a set of $2n^2$ elements. So the proof is completed. $\square$

There are still many open problems in this field. We finish with presenting some of them.

Open Problems 4.6. (1) Does the pair (c_0, c_0) satisfy the Bishop-Phelps-Bollobás property for bilinear forms?

(2) If X is uniformly convex, then $L(^nX \times \cdots \times X; Y)$ has the BPBP. Is it true for the space of bounded symmetric multilinear mappings? Recently it was shown in [**27**] that the BPBP holds for symmetric bilinear forms on a Hilbert space.

(3) Lindenstrauss [**34**] proved in 1963 that the set of operators whose second adjoints attain their norms is dense in the space of bounded linear operators between Banach spaces. That result was extended in [**8**] to multilinear mappings. However it remains open for symmetric multilinear mappings.

(4) In [**14**] it was shown that for a Banach space X the set of continuous 2-homogeneous polynomials whose canonical extensions to X^{**} attain their norms is dense. What about the case of continuous n-homogeneous polynomials ($n \geq 3$)?

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