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Restriction of the Fourier transform to some complex curves

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ABSTRACT

The purpose of this paper is to prove a Fourier restriction estimate for certain 2-dimensional surfaces in $\mathbb{R}^{2d}$, $d \geq 3$. These surfaces are defined by a complex curve $\gamma(z)$ of simple type, which is given by a mapping of the form

$$z \mapsto \gamma(z) = \left(z,\ z^2, \ldots,\ z^{d-1},\ \phi(z)\right)$$

where $\phi(z)$ is an analytic function on a domain $\Omega \subset \mathbb{C}$. This is regarded as a real mapping $z = (x, y) \mapsto \gamma(x, y)$ from $\Omega \subset \mathbb{R}^2$ to $\mathbb{R}^{2d}$.

Our results cover the case $\phi(z) = z^N$ for any nonnegative integer N, in all dimensions $d \geq 3$. The main result is a uniform estimate, valid when $d = 3$, where $\phi(z)$ may be taken to be an arbitrary polynomial of degree at most N. It is uniform in the sense that the operator norm is independent of the coefficients of the polynomial. These results are analogues of the uniform restricted strong type estimates in [5], valid for polynomial curves of simple type and some other classes of curves in $\mathbb{R}^d$, $d \geq 3$.

1. Introduction and statement of results

Let us consider a 'complex curve' of simple type in $\mathbb{C}^d$, $d \geq 2$. By this we mean a mapping of the following form:

$$z \mapsto \gamma(z) = \left(z, z^2, \ldots, z^{d-1}, \phi(z)\right), \quad z \in \Omega \tag{1.1}$$

where $\phi(z)$ is an analytic function on a domain $\Omega \subset \mathbb{C}$. We will regard this mapping as a 2-dimensional surface in $\mathbb{R}^{2d}$, given by the real mapping

$$z = (x, y) \mapsto \gamma(x, y) = \left(x, y, x^2 - y^2, 2xy, \ldots, \operatorname{Re}(\phi(z)), \operatorname{Im}(\phi(z))\right) \in \mathbb{R}^6.$$

In what follows we use $\mathbb{C}$ and $\mathbb{R}^2$ interchangeably whenever there is no danger of confusion.

Let us consider a Fourier restriction estimate of the following form:

$$\left(\int_{\mathbb{R}^2} |\widehat{f}(\gamma(z))|^q\, w(z)\, d\mu(z)\right)^{1/q} \leq C_p \|f\|_{L^p(\mathbb{R}^{2d})} \tag{1.2}$$

where $\widehat{f}(\xi)$ denotes the Fourier transform of $f \in L^p(\mathbb{R}^{2d})$, and the weight function $w(z)$ is given by

$$w(z) = |\tau(z)|^{4/(d^2+d)}, \quad \text{where } \tau(z) = \det(\gamma'(z), \ldots, \gamma^{(d)}(z)). \tag{1.3}$$

Also, $d\mu$ denotes the surface measure given by $d\mu(z) = d\mu(\gamma(z)) = dxdy$ for $z = x + iy$. Here, $\widehat{f}(\gamma(z))$ stands for $\widehat{f}(\gamma(x, y))$.

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For γ given by (1.1), we have $\tau(z) = c_d\,\phi^{(d)}(z)$ with $c_d = 2!\cdots(d-1)!$. The expression $w(z)\,d\mu(z) = |\tau(z)|^{4/(d^2+d)}d\mu(z)$ is an analogue of the affine arclength measure for real curves (*cf.* [18,19,3]). See Section 2 for the optimality of this choice of measure.

When $d = 2$, Oberlin [23] proved the following.

Theorem 1.1 (*[23]; Theorem 4 and Example 3*)**.** *Let $\gamma(z) = (z, \phi(z))$, where $\phi(z)$ is an analytic function on an open set $D \subset \mathbb{C}$. Suppose that $\phi'(z)$ and the map $(z_1, z_2) \mapsto (z_1 - z_2, \phi(z_1) - \phi(z_2))$ both have generic multiplicities at most N on D and D^2, respectively.*[1] *Then there is a constant $C_p(N) < \infty$ so that for all $f \in L^p(\mathbb{R}^4)$,*

$$\left(\int_D |\widehat{f}(\gamma(z))|^q\,|\phi''(z)|^{2/3}d\mu(z)\right)^{1/q} \le C_p(N)\|f\|_{L^p(\mathbb{R}^4)} \tag{1.4}$$

whenever $1/p + 1/(3q) = 1$, $1 \le p < 4/3$.

See [10] for a related result for some 2-dimensional surfaces in $\mathbb{R}^4$ which are not necessarily given by holomorphic functions, but which satisfy a certain nondegeneracy condition. (See also [17] for an analogous result for some k-dimensional surfaces in $\mathbb{R}^d$, where $d = 2k$.)

In this paper we obtain some positive results in higher dimensions. First let us assume that $\gamma(z)$ is in the form (1.1), where $\phi(z) = z^N$, $z \in \mathbb{C}$, for an integer $N \ge 0$.

Theorem 1.2. *Given integers $d \ge 3$ and $N \ge 0$, let $\gamma(z)$ be as in (1.1), with $\phi(z) = z^N$. Then there is a constant $C(N) < \infty$ so that for all $f \in L^{p_d,1}(\mathbb{R}^{2d})$,*

$$\left(\int_{\mathbb{R}^2} |\widehat{f}(\gamma(z))|^{p_d}\,w(z)\,d\mu(z)\right)^{1/p_d} \le C(N)\|f\|_{L^{p_d,1}(\mathbb{R}^{2d})} \tag{1.5}$$

where $w(z) = |\phi^{(d)}(z)|^{4/(d^2+d)}$ and $p_d = (d^2+d+2)/(d^2+d)$.

Moreover, there is a constant $C_p(N) < \infty$ such that

$$\left(\int_{\mathbb{R}^2} |\widehat{f}(\gamma(z))|^q\,w(z)\,d\mu(z)\right)^{1/q} \le C_p(N)\|f\|_{L^p(\mathbb{R}^{2d})} \tag{1.6}$$

whenever $1/p + 2/[(d^2+d)q] = 1$, $1 \le p < p_d$.

These estimates (as well as those in the next theorem) are expected to be optimal on the Lorentz scale of exponents, in view of the analogous results in the real case (see [3] and Theorems 1.4 and 1.5). However, this seems to be quite difficult to show in the present context, where the (real) dimension of the surface is $k = 2$. For instance, it is unknown if the estimate (1.15), which is dual to (1.6), fails for $q \le q_d$, $d \ge 3$, even when f is a bump function and we are in the nondegenerate case (with $w = 1$). This is related to the unsolved problem of determining the convergence exponent for the multi-dimensional Tarry's problem. In this connection, compare the statements of Theorem 1.3 (for $k = 1$) and Theorem 1.9 (for $k \ge 2$) in [1]. Notice that no information is available for the divergence of the integral in Theorem 1.9 (in [1]), while Theorem 1.3 (in [1]) gives the complete answer in the 1-dimensional case.

We show the sharpness of the condition $1/p + 2/[(d^2+d)q] = 1$ at the end of this section (see under the heading "A homogeneity argument"), and we also prove in Section 2 the optimality of the weight function $w(z)$, given after (1.5).

When $d = 3$, we get a uniform estimate valid for an *arbitrary* polynomial $\phi(z)$ of degree at most N.[2] This is an exact analogue of Theorem 1.5 for (real) curves, stated below.

Theorem 1.3. *For $d = 3$ and $N \ge 0$, let $\gamma(z) = (z, z^2, \phi(z))$, where $\phi(z)$ is an arbitrary polynomial of degree at most N. Then there is a constant $C(N) < \infty$, independent of the coefficients of $\phi(z)$, so that for all $f \in L^{7/6,1}(\mathbb{R}^6)$,*

$$\left(\int_{\mathbb{R}^2} |\widehat{f}(\gamma(z))|^{7/6}\,w(z)\,d\mu(z)\right)^{6/7} \le C(N)\|f\|_{L^{7/6,1}(\mathbb{R}^6)} \tag{1.7}$$

where $w(z) = |\phi'''(z)|^{1/3}$.

Moreover, there is a constant $C_p(N) < \infty$, independent of the coefficients of $\phi(z)$, such that

$$\left(\int_{\mathbb{R}^2} |\widehat{f}(\gamma(z))|^q\,w(z)\,d\mu(z)\right)^{1/q} \le C_p(N)\|f\|_{L^p(\mathbb{R}^6)}$$

whenever $1/p + 1/(6q) = 1$, $1 \le p < p_3 = 7/6$.

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1 Recall that $F : D \subset \mathbb{R}^k \to \mathbb{R}^k$ is said to have generic multiplicity N if $\mathrm{card}[F^{-1}(y)] \le N$ for almost all $y \in \mathbb{R}^k$. Here, $\mathrm{card}[E]$ denotes the cardinality of the set E.

2 It will be interesting if one can show a version of Theorem 1.3 for higher dimensions ($d \ge 4$) as well as an analogue of Theorem 1.4 for complex curves.

One can show that the weight functions $w(z)$ in (1.5) and (1.7) are sharp up to a multiplicative constant, as in the real case. See Proposition 2.1.

To put things in perspective, let us now recall some analogous earlier results for curves in $\mathbb{R}^d$.

Real curves. Let $t \mapsto \gamma(t)$ be a curve in $\mathbb{R}^d$, defined on an interval I in $\mathbb{R}$. Let us consider a Fourier restriction estimate of the following form:

$$\left(\int_I |\widehat{f}(\gamma(t))|^q\, w(t)\, dt\right)^{1/q} \le C\|f\|_{L^p(\mathbb{R}^d)} \tag{1.8}$$

where $\widehat{f}(\xi)$ denotes the Fourier transform of $f \in L^p(\mathbb{R}^d)$ and

$$w(t) = |\tau(t)|^{\frac{2}{d^2+d}}, \quad \text{with } \tau(t) = \det(\gamma'(t), \dots, \gamma^{(d)}(t)). \tag{1.9}$$

Here the measure $w(t)\,dt$ is called the 'affine arclength measure' (*cf.* [18,19,3]). We are mostly interested in *uniform* estimates for (1.8), that is, we would like to take the constant C to be uniform over given classes of curves. Also, whenever appropriate we would like to prove *global* estimates, that is, for $I = \mathbb{R}$ or $(0, \infty)$.

For the history of this problem we refer the reader to [16,3,5] and the references therein. The endpoint versions of the Fourier restriction estimates (1.8) for some classes of curves were established in [5]. We shall now describe two such results. The first concerns the case of 'monomial' curves of the form

$$t \mapsto \gamma_a(t) = (t^{a_1}, t^{a_2}, \dots, t^{a_d}), \quad 0 < t < \infty \tag{1.10}$$

where $a = (a_1, \dots, a_d)$ is a d-tuple of arbitrary real numbers. For $d \ge 2$, let $p_d = (d^2+d+2)/(d^2+d)$. The endpoint result is the following.

Theorem 1.4 ([5]). *Let $w(t)\,dt = w_a(t)\,dt$ denote the affine arclength measure for the curve (1.10), where $w(t)$ is given by (1.9) with $\gamma = \gamma_a$. Then, for $d \ge 3$, there is a constant $C(d) < \infty$ such that for all $f \in L^{p_d,1}(\mathbb{R}^d)$,*

$$\left(\int_0^\infty |\widehat{f}(\gamma_a(t))|^{p_d}\, w_a(t)\, dt\right)^{1/p_d} \le C(d)\|f\|_{L^{p_d,1}(\mathbb{R}^d)}. \tag{1.11}$$

The constant in (1.11) is uniform in the sense that it does not depend on $a_1, a_2, \dots, a_d$. We would like to point out that the versions of (1.11) fail when $d = 2$ (for $p_2 = 4/3$), even in the nondegenerate case and even when the target space is replaced by $L^1(I; w dt)$ for a finite interval I. (See [6]; see also Section 1 in [3].)

The (L^p, L^q) estimates, in the optimal range $1 \le p < p_d$, $q = 2p'/(d^2+d)$, follow by interpolating (1.11) and the (L^1, L^∞) estimate. These estimates were proved earlier in [3], following the work in [19]. (For a general result in $\mathbb{R}^2$ see, for instance, [25] and the references therein.)

Similar results have been proved for some other classes of curves including the polynomial curves of 'simple' type given by

$$\Gamma_b(t) = \big(t, t^2, \dots, t^{d-1}, P_b(t)\big), \quad t \in \mathbb{R} \tag{1.12}$$

in $\mathbb{R}^d$, where P_b is an arbitrary polynomial of degree $N \ge 0$, with the coefficients $(b_0, \dots, b_N) = b \in \mathbb{R}^{N+1}$. Namely, $P_b(t) = \sum_{j=0}^N b_j t^j$. The affine arclength measure is given by $W_b(t)\,dt$, where $W_b(t) = |\tau(t)|^{2/(d^2+d)} = |c_d P_b^{(d)}(t)|^{2/(d^2+d)}$ with $c_d = 2! \cdots (d-1)!$. The endpoint estimate in this case is the following.

Theorem 1.5 ([5]). *For $d \ge 3$, there is a constant $C(N) < \infty$ so that for all $f \in L^{p_d,1}(\mathbb{R}^d)$ and $b \in \mathbb{R}^{N+1}$,*

$$\left(\int_{-\infty}^\infty |\widehat{f}(\Gamma_b(t))|^{p_d}\, W_b(t)\, dt\right)^{1/p_d} \le C(N)\|f\|_{L^{p_d,1}(\mathbb{R}^d)}. \tag{1.13}$$

Both Theorems 1.4 and 1.5 are optimal with respect to the two Lorentz exponents occurring on both sides, if we consider them as weighted Lorentz norm estimates: $L^{p_d,1}(\mathbb{R}^d) \to L^{p_d,p_d}(w\,dt)$. In particular, the strong type (L^{p_d}, L^{p_d}) estimate fails. This fact is an easy consequence of the corresponding result in [3] for the nondegenerate case, where it was shown that $L^{p_d,1}(\mathbb{R}^d)$ was the smallest possible space and $L^{p_d,p_d}(w\,dt)$ the largest possible space on the scale of Lorentz spaces. Moreover, the weight functions w $(= w_a$ or $W_b)$ are sharp up to a multiplicative constant. (See [5,24] and Section 2.)

Remark 1.6. One can also consider general polynomial curves of the form $\gamma(t) = (P_1(t), \dots, P_d(t))$, where each P_j is a polynomial of degree at most N. Dendrinos and Wright [14] established the uniform Jacobian estimate for the mapping $(t_1, \dots, t_d) \mapsto \sum_{j=1}^d \gamma(t_j)$. This implies a uniform restriction estimate in the reduced range $1 \le p < p_c(d) = \frac{d^2+2d}{d^2+2d-2}$. (This range is commonly referred to as 'Christ's range' of exponents.) This is the range where one does not need the 'method of offspring curves', hence the torsion bound is not needed here. In [5] (see Proposition 8.1 there) this range was extended a little by combining an argument of Drury [16] with a result of Stovall [27] on averaging operators.

The main obstacle for obtaining a uniform estimate in the full range, by means of the method of offspring curves, is that the second crucial estimate concerning the *torsion* of the offspring curves (as described in the beginning of Section 6) breaks down for curves of non-simple type. At the moment the only known approach that gives the full range $1 \le p < p_d$ (and also the restricted strong type for $p = p_d$) for curves of non-simple type is the method based on 'exponential parametrization', which originated in [19] and was used in [5] to prove Theorem 1.4. (See also [13] and the remark at the end of Section 6 of [5].)

The dual estimate of (1.5). Let p' denote the Hölder conjugate exponent, i.e. $1/p + 1/p' = 1$. The dual estimate of (1.5) is the following weak type (q_d, q_d) estimate for $q_d = p'_d = (d^2 + d + 2)/2$:

$$\|Tf\|_{L^{q_d,\infty}(\mathbb{R}^{2d})} \le C(N)\|f\|_{L^{q_d}(w d\mu)} \tag{1.14}$$

where T is given by

$$Tf(x) = \int_{\mathbb{R}^2} e^{ix\cdot\gamma(z)} f(z)w(z)d\mu(z), \quad x \in \mathbb{R}^{2d}.$$

Recall that the mapping $z \mapsto \gamma(z)$ is regarded as a 2-dimensional surface $(x, y) \mapsto \gamma(x, y)$ in $\mathbb{R}^{2d}$. In particular, $x \cdot \gamma(z)$ denotes the dot product in $\mathbb{R}^{2d}$.

By interpolating (1.14) with the (L^1, L^∞) estimate it follows that

$$\|Tf\|_{L^{q}(\mathbb{R}^{2d})} \le C_q(N)\|f\|_{L^{p}(w d\mu)} \tag{1.15}$$

for $1/p + (d^2 + d)/(2q) = 1$, $q > q_d = p'_d = (d^2 + d + 2)/2$.

A homogeneity argument. To see the necessity of the condition $1/p + (d^2 + d)/(2q) = 1$ for (1.15) or (1.14) to hold, we use the usual homogeneity argument. That is, we take $f = \chi_{B_R}$, where $B_R = B(0, R)$ is a ball in $\mathbb{R}^2$. We see that

$$|Tf(x)| \ge cR^{\frac{4(N-d)}{(d^2+d)}+2}\chi_{E_R}(x/a)$$

for some small constant $a > 0$, where $E_R = [-R^{-1}, R^{-1}]^2 \times [-R^{-2}, R^{-2}]^2 \times \cdots \times [-R^{-(d-1)}, R^{-(d-1)}]^2 \times [-R^{-N}, R^{-N}]^2$. Hence, if (1.14) or (1.15) holds, then we must have

$$R^{\frac{4(N-d)}{d^2+d}+2}R^{-\frac{2}{q}(\frac{d(d-1)}{2}+N)} \le CR^{(\frac{4(N-d)}{d^2+d}+2)\frac{1}{p}}, \quad \forall R > 0.$$

Thus, it follows that $1/p + (d^2 + d)/(2q) = 1$.

Organization of this paper. The optimality of the weight function $w(z)$ in Theorem 1.2 or Theorem 1.3 is proved in Section 2. Section 3 contains the proof of a lower bound for a Jacobian arising in the proof of Theorem 1.2. A uniform lower bound for the Jacobian associated to curves of simple type with arbitrary polynomials $\phi(z)$ is proved in Section 4. There is also a short discussion about a sublevel set estimate for the complex Vandermonde determinant at the end of Section 4. In Section 5 we state an interpolation theorem proved in [5]. Theorem 1.3 is proved in Section 6. Finally, in Section 7 we indicate how to modify the latter argument to prove Theorem 1.2.

Notation. Adopting the usual convention, we let C or c represent strictly positive constants whose values may not be the same at each occurrence. These constants may usually depend on N, d and p, but they will always be independent of f. (In addition, they are uniform over the class of $\gamma(z)$ given in Theorem 1.3. In particular, they are independent of the coefficients of the polynomial $\phi(z)$ throughout the proof of that result.) Their dependence on the parameters is sometimes indicated by a subscript or shown in parentheses. We write $A \lesssim B$ or $B \gtrsim A$ to mean $A \le CB$, and $A \approx B$ means both $A \lesssim B$ and $B \lesssim A$.

2. Optimality of the weight function

Let $d \ge 2$. Here we shall consider the more general mapping $\gamma(z) = (\phi_1(z), \dots, \phi_d(z))$, where each ϕ_j is an analytic function on $\Omega \subset \mathbb{C}$. We continue to use the notation $\tau(z) = \det(\gamma'(z), \dots, \gamma^{(d)}(z))$. The following result is analogous to the one found in Section 2 of [5], which in turn is based on an argument in [24].

Proposition 2.1. *Assume that for some $p \in (1, p_d]$ and $q(p) = 2p'/(d^2 + d)$ there is a constant B such that for all $f \in L^{p,1}(\mathbb{R}^{2d})$,*

$$\left(\int_\Omega |\widehat{f}(\gamma(z))|^{q(p)}\,\omega(z)\,d\mu(z)\right)^{1/q(p)} \le B\|f\|_{L^{p,1}(\mathbb{R}^{2d})} \tag{2.1}$$

where $\omega(z)$ is a nonnegative, locally integrable weight function on Ω. Then there is a constant C_d such that

$$\omega(z) \le C_d\, B^{q(p)}|\tau(z)|^{\frac{4}{d^2+d}} \quad a.e.\ z \in \Omega. \tag{2.2}$$

When $\gamma(z)$ is as in (1.1), then we have $\tau(z) = c_d\,\phi^{(d)}(z)$, so that the last inequality becomes $\omega(z) \le C_d\, B^{q(p)} |\phi^{(d)}(z)|^{4/(d^2+d)}$, as we wanted to show.

Proof. Let $P = AQ + b$ be a parallelepiped in $\mathbb{R}^{2d}$, where $Q = [-\frac{1}{2}, \frac{1}{2}]^{2d}$, $b \in \mathbb{R}^{2d}$ and A is an invertible linear transformation on $\mathbb{R}^{2d}$. Take $\widehat{f}(\xi) = \exp(-\pi|A^{-1}(\xi - b)|^2)$. Then $|\widehat{f}(\xi)| \ge c_0 > 0$ for $\xi \in P$, and $f(x) = e^{2\pi i b\cdot x}|\det(A)| \cdot \exp(-\pi|A^t x|^2)$.

Since $|P| = |\det(A)|$, we have $\|f\|_{p,1} \approx |P|^{1/p'}$. Hence, (2.1) implies that

$$\int_{\mathbb{R}^2} \chi_P(\gamma(z))\, \omega(z)\, d\mu(z) \le C(d)\, B^{q(p)} |P|^{2/(d^2+d)}. \tag{2.3}$$

Since each $\phi_j(z)$ is analytic on Ω, so is $\tau(z)$. Thus, we may assume $\tau(z)$ has only isolated zeros. So, it is enough to show (2.2) at points where $\tau(z) \neq 0$. (Otherwise, $\tau(z)$ is identically zero. We comment on this case at the end of this section.)

Fix $a \in \Omega$. We have

$$\gamma(a+z) = \gamma(a) + \sum_{j=1}^{d} \frac{z^j}{j!} \gamma^{(j)}(a) + O(|z|^{d+1}) \tag{2.4}$$

for z near the origin. Now consider the linear mapping

$$(z_1, \dots, z_d) \mapsto \Phi(z_1, \dots, z_d) = \gamma(a) + \sum_{j=1}^{d} \frac{z_j}{j!} \gamma^{(j)}(a). \tag{2.5}$$

Write $z_j = x_j + iy_j$. For $\varepsilon > 0$, let $E = \{(z_1, \dots, z_d) :\ |x_j| \le 2\,\varepsilon^j, |y_j| \le 2\,\varepsilon^j, 1 \le j \le d\}$ denote a rectangular box in $\mathbb{R}^{2d}$. The image P_1 of E under this mapping is a parallelepiped in $\mathbb{R}^{2d}$. Its volume $|P_1|$ is given by

$$\begin{aligned} |P_1| &= 2^{2d}\, \varepsilon^{d^2+d} \cdot J_{\mathbb{R}}\Phi = 2^{2d}\, \varepsilon^{d^2+d} \cdot |\det J_{\mathbb{C}}\Phi|^2 \\ &= 2^{2d}\, \varepsilon^{d^2+d} \cdot |(2! \cdots d!)^{-1} \det(\gamma'(a), \dots, \gamma^{(d)}(a))|^2 \\ &= 2^{2d} (2! \cdots d!)^{-2}\, \varepsilon^{d^2+d} \cdot |\tau(a)|^2. \end{aligned}$$

We used here the fact that the Jacobian of (2.5) as a real mapping is given by $J_{\mathbb{R}}\Phi = |\det J_{\mathbb{C}}\Phi|^2$, where $J_{\mathbb{C}}\Phi$ is the holomorphic Jacobian matrix of the mapping (2.5). This is a consequence of Proposition 1.4.10 on p. 51 in [22].

If $\tau(a) \neq 0$, and if $\varepsilon = \varepsilon(a) > 0$ is sufficiently small, then we have $\gamma(a+z) \in P_1$ when $|z| \le \varepsilon$. In fact, since $\gamma'(a), \dots, \gamma^{(d)}(a)$ span $\mathbb{C}^d$, it follows from (2.4) that

$$\gamma(a+z) = \gamma(a) + \sum_{j=1}^{d} \frac{z^j + z^d g_j(z,a)}{j!} \gamma^{(j)}(a) \tag{2.6}$$

for some functions $g_j(z,a)$ such that $g_j(z,a) \to 0$ as $z \to 0$ for $j = 1, 2, \dots, d$.

Therefore, it follows from (2.3) that

$$\limsup_{\varepsilon \to 0} \frac{1}{\pi \varepsilon^2} \int_{|z| \le \varepsilon} \omega(a+z)\, d\mu(z) \le C_d\, B^{q(p)} |\tau(a)|^{4/(d^2+d)}.$$

So the conclusion (2.2) follows by the Lebesgue differentiation theorem.

On the other hand, when $\tau(a) = 0$, a slight modification of the above argument shows that

$$\int_{|z| \le \varepsilon} \omega(a+z)\, d\mu(z) = o(\varepsilon^2), \quad \text{as } \varepsilon \to 0.$$

Thus, when $\tau(z) \equiv 0$, we may conclude that $\omega(z)$ is zero almost everywhere. (See Section 2 of [5] for more details.) $\square$

3. A lower bound for the Jacobian

Let us begin by making a definition.

Definition 3.1. Let N be a nonnegative integer and let $z_1, \dots, z_d$ be complex numbers. Let Q_N denote a homogeneous monic polynomial of degree N in $z_1, \dots, z_d$, given by

$$Q_N(z_1, \dots, z_d) = \sum_{\alpha_1 + \cdots + \alpha_d = N} z_1^{\alpha_1} \cdots z_d^{\alpha_d}.$$

Here, $\alpha_1, \dots, \alpha_d$ are nonnegative integers.

Thus, Q_N is a *symmetric* polynomial. We have the following properties of Q_N.

Lemma 3.2. *Let $d \ge 2$ and $N \ge 1$. Then*

(i) $Q_0(z_d, \dots, z_1) = 1$;

(ii) $Q_N(z_3, z_1) - Q_N(z_2, z_1) = (z_3 - z_2) Q_{N-1}(z_3, z_2, z_1)$;

(iii) $Q_N(z_d, z_{d-1}, \dots, z_1) = Q_N(z_d, \dots, z_2) + Q_{N-1}(z_d, \dots, z_2) z_1 + \cdots + Q_1(z_d, \dots, z_2) z_1^{N-1} + z_1^N$.

(iv) *Moreover, we have*

$$Q_N(z_{d+1}, z_{d-1}, \dots, z_1) - Q_N(z_d, z_{d-1}, \dots, z_1) = (z_{d+1} - z_d)\, Q_{N-1}(z_{d+1}, \dots, z_1).$$

Proof. The properties (i)–(iii) are straightforward. To see that (iv) holds, we use induction on d. First, (ii) gives the case $d = 2$. Now suppose that (iv) holds with d replaced by $d - 1$. That is, we assume

$$Q_N(z_d, z_{d-2}, \dots, z_1) - Q_N(z_{d-1}, z_{d-2}, \dots, z_1) = (z_d - z_{d-1})Q_{N-1}(z_d, \dots, z_1)$$

holds for some $d \geq 3$ and for $N \geq 1$. It follows from (iii) and this induction hypothesis that

$$\begin{aligned} Q_N(z_{d+1}, z_{d-1}, \dots, z_1) - Q_N(z_d, z_{d-1}, \dots, z_1) &= Q_N(z_{d+1}, z_{d-1}, \dots, z_2) + Q_{N-1}(z_{d+1}, z_{d-1}, \dots, z_2)z_1 + \cdots + z_1^N \\ &\quad - [Q_N(z_d, z_{d-1}, \dots, z_2) + Q_{N-1}(z_d, z_{d-1}, \dots, z_2)z_1 + \cdots + z_1^N] \\ &= (z_{d+1} - z_d)\big[Q_{N-1}(z_{d+1}, z_d, \dots, z_2) + Q_{N-2}(z_{d+1}, z_d, \dots, z_2)z_1 \\ &\quad + \cdots + Q_1(z_{d+1}, z_d, \dots, z_2)z_1^{N-2} + z_1^{N-1}\big] \\ &= (z_{d+1} - z_d)\, Q_{N-1}(z_{d+1}, \dots, z_1) \end{aligned}$$

which is the case d of (iv). Hence, (iv) holds for all $d \geq 2$ and $N \geq 1$.

We now turn to the proof of a lower bound for the Jacobian of a transformation that arises in the proof of Theorem 1.2. Let $J(z_1, \dots, z_d) = J_{\mathbb{C}}(z_1, \dots, z_d)$ denote the determinant of the holomorphic Jacobian matrix of the mapping

$$(z_1, \dots, z_d) \mapsto \mathbf{\Gamma}(z_1, \dots, z_d) = \sum_{k=1}^{d} \Gamma_b(z_k)$$

with $z_k = z + h_k$, $h_1 = 0$. Here, $\Gamma_b(z) = m^{-1}\sum_{j=1}^{m} \gamma(z + b_j)$, where $m \in \mathbb{N}$, and $b = (b_1, \dots, b_m) \in \mathbb{C}^m$, with $b_1 = 0$.

For $h = (h_1, \dots, h_d) \in \mathbb{C}^d$, let $\mathbf{v}(h) = \mathbf{v}(h_1, \dots, h_d) = \prod_{1\leq i<j\leq d} |h_i - h_j|$ denote the absolute value of the (complex) Vandermonde determinant.

Lemma 3.3. *Let $\gamma(z)$ be given by (1.1) with $\phi(z) = z^N$ for an integer $N \geq d$ with $d \geq 2$, and let $\Gamma_b(z)$ be defined as above. Set*

$$J(z_1, \dots, z_d) = J_{\mathbb{C}}(z_1, \dots, z_d) = \det(\Gamma_b'(z_1), \dots, \Gamma_b'(z_d))$$

where $z_j = z + h_j \in \mathbb{C}$, $1 \leq j \leq d$, and $h_1 = 0$. Then $\mathbb{C}$ may be written as the union (ignoring a null-set) of $C(d, N)$ sectors Δ_ℓ with vertex at the origin such that for each $1 \leq \ell \leq C(d, N)$, and for each integer $m \geq 1$, we have

$$|J(z_1, \dots, z_d)| \geq c(d, N)\, \mathbf{v}(h)\, \max\left\{ \frac{1}{m} \sum_{j=1}^{m} |\phi^{(d)}(z + b_j + h_k)| : 1 \leq k \leq d \right\} \tag{3.1}$$

where $z + b_j + h_k \in \Delta_\ell$. Here, $C(d, N)$ and $c(d, N)$ are positive constants depending only on d and N.

Note that $\mathbf{v}(h) = \mathbf{v}(z_1, \dots, z_d)$, since $z_i - z_j = h_i - h_j$.

Proof. Let us write $z_{jk} = z + b_j + h_k$. Since $h_1 = 0$, we have $z_{j1} = z + b_j$. If we abbreviate $\sum_{j=1}^{m}$ as $\sum$, we get

$$\begin{aligned} J(z_1, \dots, z_d) &= \det(\Gamma_b'(z + h_1), \dots, \Gamma_b'(z + h_d)) \\ &= \frac{(d-1)!N}{m^{d-1}} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \sum(z_{j1}) & \sum(z_{j2}) & \cdots & \sum(z_{jd}) \\ \sum(z_{j1})^2 & \sum(z_{j2})^2 & \cdots & \sum(z_{jd})^2 \\ \vdots & \vdots & \ddots & \vdots \\ \sum(z_{j1})^{d-2} & \sum(z_{j2})^{d-2} & \cdots & \sum(z_{jd})^{d-2} \\ \sum(z_{j1})^{N-1} & \sum(z_{j2})^{N-1} & \cdots & \sum(z_{jd})^{N-1} \end{vmatrix} \\ &= \frac{(d-1)!N}{m^{d-1}} \begin{vmatrix} 1 & 0 & \cdots & 0 \\ \sum(z_{j1}) & mh_2 & \cdots & mh_d \\ \sum(z_{j1})^2 & h_2 \sum Q_1(z_{j2}, z_{j1}) & \cdots & h_d \sum Q_1(z_{jd}, z_{j1}) \\ \vdots & \vdots & \ddots & \vdots \\ \sum(z_{j1})^{d-2} & h_2 \sum Q_{d-3}(z_{j2}, z_{j1}) & \cdots & h_d \sum Q_{d-3}(z_{jd}, z_{j1}) \\ \sum(z_{j1})^{N-1} & h_2 \sum Q_{N-2}(z_{j2}, z_{j1}) & \cdots & h_d \sum Q_{N-2}(z_{jd}, z_{j1}) \end{vmatrix}. \end{aligned}$$

Note that the value of this determinant equals

$$\begin{vmatrix} mh_2 & mh_3 & \cdots & mh_d \\ h_2\sum Q_1(z_{j2},z_{j1}) & h_3\sum Q_1(z_{j3},z_{j1}) & \cdots & h_d\sum Q_1(z_{jd},z_{j1}) \\ h_2\sum Q_2(z_{j2},z_{j1}) & h_3\sum Q_2(z_{j3},z_{j1}) & \cdots & h_d\sum Q_2(z_{jd},z_{j1}) \\ \vdots & \vdots & \ddots & \vdots \\ h_2\sum Q_{d-3}(z_{j2},z_{j1}) & h_3\sum Q_{d-3}(z_{j3},z_{j1}) & \cdots & h_d\sum Q_{d-3}(z_{jd},z_{j1}) \\ h_2\sum Q_{N-2}(z_{j2},z_{j1}) & h_3\sum Q_{N-2}(z_{j3},z_{j1}) & \cdots & h_d\sum Q_{N-2}(z_{jd},z_{j1}) \end{vmatrix} = mh_2h_3\cdots h_d$$

$$\times\begin{vmatrix} 1 & 0 & \cdots & \cdots & 0 \\ \sum Q_1(z_{j2},z_{j1}) & m(h_3-h_2) & \cdots & \cdots & m(h_d-h_2) \\ \sum Q_2(z_{j2},z_{j1}) & (h_3-h_2)\sum Q_1(z_{j3},z_{j2},z_{j1}) & \cdots & \cdots & (h_d-h_2)\sum Q_1(z_{jd},z_{j2},z_{j1}) \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \sum Q_{d-3}(z_{j2},z_{j1}) & (h_3-h_2)\sum Q_{d-4}(z_{j3},z_{j2},z_{j1}) & \cdots & \cdots & (h_d-h_2)\sum Q_{d-4}(z_{jd},z_{j2},z_{j1}) \\ \sum Q_{N-2}(z_{j2},z_{j1}) & (h_3-h_2)\sum Q_{N-3}(z_{j3},z_{j2},z_{j1}) & \cdots & \cdots & (h_d-h_2)\sum Q_{N-3}(z_{jd},z_{j2},z_{j1}) \end{vmatrix}$$

by the properties of Q_N stated in Lemma 3.2.

Continuing in this way, we see that

$$\begin{aligned} J(z_1,\ldots,z_d) &= (d-1)!Nm^{-1}(h_2\cdots h_d)\cdots(h_{d-1}-h_{d-2})(h_d-h_{d-2}) \\ &\quad\times\begin{vmatrix} 1 & 1 \\ \sum Q_{N-d+1}(z_{j,d-1},z_{j,d-2},\ldots,z_{j1}) & \sum Q_{N-d+1}(z_{j,d},z_{j,d-2},\ldots,z_{j1}) \end{vmatrix} \\ &= \frac{(d-1)!N}{m}\prod_{1\le k<l\le d}(h_l-h_k)\sum_{j=1}^{m} Q_{N-d}(z_{j,d},z_{j,d-1},\ldots,z_{j1}). \end{aligned}$$

Hence, if we write L_j for $Q_{N-d}(z_{jd},\ldots,z_{j1})$, we obtain

$$|J(z_1,\ldots,z_d)| \ge \frac{(d-1)!N}{m}\mathbf{v}(h)\cdot\left|\sum_{j=1}^{m} L_j\right|.$$

By rotation, it suffices to consider the case when $\Delta_\ell = \Delta = \{re^{i\theta} : r \ge 0, \text{ and } \theta \in (0,\varepsilon)\}$, for some small $\varepsilon = \varepsilon(d,N) > 0$. (Indeed, we may express the elements of Δ_ℓ in the form $z' = az$, for $z \in \Delta$ and some fixed complex number a with $|a| = 1$. By homogeneity, the powers of a may be factored out of each row of the Jacobian.)

Recalling that $z_{jk} = z + b_j + h_k$, let us write $x_{jk} = \mathrm{Re}(z_{jk})$ and $y_{jk} = \mathrm{Im}(z_{jk})$. Then for each j, we have the lower bound

$$|\mathrm{Re}(L_j)| \ge Q_{N-d}(x_{j1},x_{j2},\ldots,x_{jd}) + E_j$$

where E_j is a sum of $C(d,N)$ terms similar to the expression preceding it but with one or more factors x_{jk} replaced by $c_{jk}\,y_{jk}$. Here, $|c_{jk}| \le C'(d,N)$. Note that $0 < y_{jk} \lesssim \varepsilon\, x_{jk}$. Hence the last expression is bounded below by

$$\frac{1}{2}Q_{N-d}(x_{j1},x_{j2},\ldots,x_{jd}) \gtrsim \sum_{k=1}^{d} x_{jk}^{N-d} \approx \sum_{k=1}^{d} |\phi^{(d)}(z+b_j+h_k)|$$

provided that $\varepsilon = \varepsilon(d,N) > 0$ is chosen sufficiently small. This implies that

$$|J(z_1,\ldots,z_d)| \ge c(d,N)\,\mathbf{v}(h)\,\frac{1}{m}\sum_{k=1}^{d}\sum_{j=1}^{m}|\phi^{(d)}(z+b_j+h_k)|$$

whenever $z + b_j + h_k \in \Delta$. This finishes the proof of Lemma 3.3. □

4. Jacobian bound for polynomial curves of simple type in $\mathbb{C}^3$

A version of the following lemma may be found in [20] (Lemma 3.1), where it is stated and proved for polynomials of a real variable. (See also [8,9].) But the same proof works for polynomials of a complex variable, since it only relies on the triangle inequality.

Lemma 4.1. *Given a complex number $D \neq 0$, let $P(z) = D\prod_{j=1}^{N}(z - z_j) = \sum_{k=0}^{N} \nu_k z^k$ be a polynomial of degree N. Assume that the roots z_j are ordered so that $|z_1| \leq \cdots \leq |z_N|$. Let $G_j = \{z \in \mathbb{C} : A|z_j| \leq |z| \leq A^{-1}|z_{j+1}|\}$ for $1 \leq j \leq N-1$, and $G_N = \{z \in \mathbb{C} : |z| \geq A|z_N|\}$. Then there exists a constant $C = C(N) > 1$ such that for any $A \geq C(N)$ and $1 \leq j \leq N$, if G_j is nonempty, then*

(i) *$|P(z)| \approx |\nu_j||z|^j$ for $z \in G_j$;*

(ii) *for $1 \leq j \leq N-1$, we have $|\nu_j| \approx |D|\prod_{\ell=j+1}^{N}|z_\ell|$. (For $j = N$, we have $\nu_N = D$. In particular, $\nu_j \neq 0$, $1 \leq j \leq N$.)*

The idea of this lemma helps us prove a uniform lower bound, i.e. Lemma 4.2, for the Jacobian associated to complex curves of simple type in $\mathbb{C}^3$, when $\phi(z)$ is an arbitrary polynomial. This result may be of some independent interest. For instance, it is likely to have some implications for the related averaging operators. (See e.g. [12,27].)

Lemma 4.2. *Let $\gamma(z) = (z, z^2, \ldots, z^{d-1}, \phi(z))$, where $\phi(z)$ is a polynomial of degree at most N. Let $J(z_1, \ldots, z_d) = J_{\mathbb{C}}(z_1, \ldots, z_d)$ be the determinant of the holomorphic Jacobian of the transformation $(z_1, \ldots, z_d) \mapsto \sum_{i=1}^{d}\gamma(z_i)$.*

If $d = 3$, then there exist a constant $c(d, N) > 0$, a positive integer $M = M(d, N)$, and a collection of pairwise disjoint, convex open sets $B_1, \ldots, B_M$, such that $\mathbb{C} = \cup_{\ell=1}^{M} B_\ell$, ignoring a null-set, and such that for $1 \leq \ell \leq M$,

$$|J(z_1, \ldots, z_d)| \geq c(N)\,\mathbf{v}(z_1, \ldots, z_d) \max_{1\leq i\leq d}|\phi^{(d)}(z_i)| \tag{4.1}$$

whenever $z_j \in B_\ell$, $1 \leq j \leq d$.

Recall that $\mathbf{v}(z_1, \ldots, z_d) = \prod_{1\leq i<j\leq d}|z_i - z_j|$. Thus, $\mathbf{v}(z_1, z_2, z_3) = |z_1 - z_2| \cdot |z_1 - z_3| \cdot |z_2 - z_3|$, when $d = 3$.

Remark 4.3. If $\gamma(z)$ in Lemma 4.2 is replaced by (an offspring curve)

$$\Gamma(z) = (P_1(z), \ldots, P_{d-1}(z), \phi(z))$$

where $P_j(z) = z^j +$ lower order terms as in (6.1), then the Jacobian of the corresponding mapping is the same as that for $\gamma(z)$ when they have the same $\phi(z)$. So, we should obtain the same conclusion (4.1) in this case. For example, when $d = 3$, the new Jacobian $J(z_1, z_2, z_3)$ is again given by the formula (4.6).

Proof of Lemma 4.2. Let $d = 3$. If $0 \leq N \leq 2$, then $\phi''' \equiv 0$ and $J \equiv 0$. Moreover, if $N = 3$, then $\phi'''(z)$ is a non-zero constant and $J(z_1, z_2, z_3)$ is a constant multiple of $\mathbf{v}(z_1, z_2, z_3)$. Thus, we may assume that $N \geq 4$ and $\phi'''(z)$ has at least one zero. Our goal is to decompose $\mathbb{C}$ into a collection $\{B\}$ of $M(N)$ pairwise disjoint, convex open sets so that the inequality (4.1) holds on each B. To this end, we will represent $J(z_1, z_2, z_3)$ as an integral as in (4.6). It may be worthwhile to point out that, compared to the real case, the complex case is more delicate, because it is necessary to control carefully the argument of the integrand as well as the magnitude, in order to get a good lower bound for the multiple integral of a function of a complex variable.

For the sake of clarity we will divide the rest of the proof into four steps.

Step 1. *Preliminary decompositions of* $\mathbb{C}$.

To get a decomposition of $\mathbb{C}$, we begin by fixing a zero b of $\phi'''(z)$, putting $S = S(b) = \{z \in \mathbb{C} : |z-b| < |z-b'|, \forall b' \neq b\}$ as in [14], where $\{b'\}$ is the zero set of $\phi'''(z)$. Then $\mathbb{C} = \bigcup_{b:\phi'''(z)=0} S(b)$, except for a null-set. We will show how to decompose $S(b)$ further in four different ways.

By a translation, we may assume that $b = 0$. Let us write $\phi'''(z) = Dz^{a_1}\prod_{j=2}^{m}(z - \eta_j)^{a_j}$, where 0 and η_j are the distinct roots of $\phi'''(z)$, with multiplicity a_j, so that $N - 3 = a_1 + \cdots + a_m$.

(1.a) *Decomposition into gap annuli and dyadic annuli.* Let us rewrite $\phi'''(z) = D\prod_{j=1}^{N}(z-z_j) = \sum_{k=0}^{N}\nu_k z^k$ as in Lemma 4.1, with z_j, ν_j and G_j as in that lemma. (By abuse of notation we will write N, instead of $N-3$, for $\deg(\phi''')$. Thus, we have $N \geq 1$ in this new notation.) Since a constant factor in $\phi(z)$ can be canceled from both sides of the inequality (4.1), we may assume that $D = 1$. Since $\phi'''(0) = 0$, we have $z_1 = 0$. The region G_j may be called a 'gap annulus' in analogy with the terminology 'gap interval' in [14]. From Lemma 4.1 it follows that $|\phi'''(z)| \approx |\nu_j||z|^j$ for $z \in G_j$. Also, define the 'dyadic annuli' by

$$D_j = \{z \in \mathbb{C} : A_1^{-1}|z_j| < |z| < A_1|z_j|\}, \quad 2 \leq j \leq N-1,$$

for some $A_1 > 0$ chosen slightly larger than A. Thus, there is a small overlap between the regions G_j and D_j, which will help us define certain *convex* open sets B contained in them, cutting off some parts of the non-convex regions (annuli) G_j and D_j. (See the second paragraph under the heading 'Decomposition of G_j' below.)

(1.b) *Decomposition into sectors.* By dividing $\mathbb{C}$ into narrow sectors $\{\Delta\}$ with vertex at 0, and then by using rotation, we may assume $0 < y < \varepsilon x$ in Δ, for some $\varepsilon = \varepsilon(N)$, where we have written $z = x + iy$. (See the proof of Theorem 3.3.) Then we have $|\phi'''(z)| \approx |\nu_j| \cdot |z|^j \approx |\nu_j| \cdot x^j$, for $z \in \Delta \cap G_j$.

Step 2. *Further decompositions of the regions.*

Fix j and let $z \in \Delta \cap G_j$. Recall that $\phi'''(z) = \prod_{j=1}^{N}(z - z_j)$ with $z_1 = 0$. Let us rewrite it in the form

$$\phi'''(z) = g(z)(-1)^{N-j} z^j \prod_{\ell=j+1}^{N} z_\ell \tag{4.2}$$

where

$$g(z) = \prod_{i=1}^{j}\left(1 - \frac{z_i}{z}\right) \prod_{\ell=j+1}^{N}\left(1 - \frac{z}{z_\ell}\right).$$

We want to decompose the range of $g(z)$, contained in an annulus, into small radial sectors. By considering the pre-images of the sectors we want to decompose $S \cap \Delta \cap G_j$ and $S \cap \Delta \cap D_j$ further into convex sets $\{B\}$ with the following property.

After multiplying by a unit complex number if necessary, $g(z)$ can be put in the form $g(z) = \xi(z) + i\eta(z)$ with

$$0 < b_0 \,|\eta(z)| \le \xi(z) \tag{4.3}$$

for all $z \in B \subset S \cap \Delta \cap E_j$ (with $E_j = G_j$ or D_j), where $b_0 > 0$ is a large absolute constant to be chosen later. If this holds, then we have $\xi(z) \le |g(z)| \le (1 + b_0^{-2})^{1/2}\xi(z)$ for $z \in B$.

To achieve this goal, we need to decompose G_j and D_j further. This can be done separately for G_j and D_j as follows.

(2.a) *Decomposition of* G_j. If $z \in S \cap \Delta \cap G_j$, we have $A|z_j| \le |z| \le |z_{j+1}|/A$. We may assume $z_{j+1} \ne 0$, since otherwise $G_j = \{0\}$ and there is nothing to prove. Since $1 - z_i/z = 1 + O(1/A)$, $1 \le i \le j$, and also $1 - z/z_\ell = 1 + O(1/A)$, $j+1 \le \ell \le N$, taking $A = C_0N$ gives $g(z) = 1 + O(C_0^{-1})$. In fact, it is easy to see that $|g(z) - 1| \le 2\,C_0^{-1}$, which yields the condition (4.3) if we choose $C_0 \ge 3\,b_0$, say.

It only remains to cut $S \cap \Delta \cap G_j$ into a few *convex* open sets B so that their union covers all of $S \cap \Delta \cap G_j$, except for a null set and some little pieces which lie in the intersections $D_i \cap G_j \cap S \cap \Delta$, for $i = j$ and $i = j + 1$. (The remaining parts of the sets $D_i \cap G_j \cap S \cap \Delta$, for $i = j,\ j + 1$, will be covered by the B's arising from the decomposition of D_i, which is described next.)

(2.b) *Decomposition of* D_j. If $z \in S \cap \Delta \cap D_j$, we have $A_1^{-1}|z_j| < |z| < A_1|z_j|$, where $A_1 = (1+\delta_0)A = C_1N = (1+\delta_0)C_0N$ for some small $\delta_0 > 0$. (Recall that the purpose of introducing $A_1 > A$ is to create some overlap between the regions ($S_1 \cap \Delta \cap D_j$ and $S_1 \cap \Delta \cap G_j$) in order to facilitate the decomposition of $\mathbb{C}$ into convex regions. It would be difficult to decompose annular regions into convex regions.)

We may assume $z_j \ne 0$ here, since otherwise D_j is empty. Let us recall $\phi'''(z) = g(z)(-1)^{N-j} z^j \prod_{\ell=j+1}^{N} z_\ell$, as in (4.2).

Note that $|(z - z_i)/z| \ge 1$ for all i if $z \in S$, and also $|(z_\ell - z)/z_\ell| \ge (1/2)$ for all ℓ if $z \in S$. In fact, the second inequality follows from the first, since $|z_\ell| \le |z - z_\ell| + |z| \le 2|z - z_\ell|$ if $z \in S$. From this it follows that

$$|g(z)| \ge 2^{j-N} \ge 2^{2-N} \quad \forall z \in S \cap \Delta \cap D_j,\ 2 \le j \le N. \tag{4.4}$$

The inequality (4.4) gives a separation from the origin, which is needed to obtain a small angular support for $g(B)$ so that (4.3) holds, where B is to be specified shortly.

Moreover, we have $|\partial_r(1 - z_i/z)| \le |z_i|/r^2 \le |z_j|/r^2 \le A_1^2/|z_j|$ (for $i \le j$) and $|\partial_r(1 - z/z_\ell)| \le 1/|z_\ell| \le 1/|z_j|$ (for $\ell \ge j$). Hence,

$$|\partial_r(g(r,\theta))| \le N(1 + A_1)^{N+1}|z_j|^{-1}.$$

Likewise, we get $|\partial_\theta(1 - z_i/z)| \le |z_i|/r \le |z_j|/r \le A_1$ (for $i \le j$) and $|\partial_\theta(1 - z/z_\ell)| \le r/|z_\ell| \le r/|z_j| \le A_1$ (for $\ell \ge j$). So, $|\partial_\theta(g(r,\theta))| \le N(1 + A_1)^N$.

Hence, we can divide the r-interval, given by $A_1^{-1}|z_j| < r < A_1|z_j|$, into $C(N)$ pieces of length $L \le C(N)^{-1}A_1|z_j|$ so that

$$\begin{aligned} |\partial_r(g(r,\theta))| \cdot L &\le N(1 + A_1)^{N+1}|z_j|^{-1} \times C(N)^{-1}A_1|z_j| \\ &\le C(N)^{-1}N(1 + A_1)^{N+2}. \end{aligned} \tag{4.5}$$

(Note that the two factors involving $|z_j|$ cancel out.)

Similarly, if we divide the θ-interval into $C(N)$ pieces of angle Θ, then we have $|\partial_\theta(g(r,\theta))| \cdot \Theta \lesssim N(1 + A_1)^N \times \varepsilon(N)\,C(N)^{-1}$. Since this is smaller than the previous estimate, for simplicity we can use the same number $C(N)$ here.

This allows us to choose $C(N)^2$ pairwise disjoint, convex open sets $\{B\}$ in $S \cap \Delta \cap D_j$ such that $g(B)$ is contained in a small disk of diameter $\lesssim C(N)^{-1}N(1 + A_1)^{N+2}$. We can do this in such a way that the collection $\{B\}$, which consists of all the B's from this step (for D_j, $2 \le j \le N$) and the previous one (for G_j, $1 \le j \le N$), covers all of $S \cap \Delta$, except for a null-set.

The estimates (4.4) and (4.5) imply that the angular support of $g(B)$ (when the angle is measured from 0) is bounded by

$$\frac{C_2\, C(N)^{-1} N(1+A_1)^{N+2}}{c_0\, 2^{2-N}} = \frac{C_2\, 2^N N(1+C_1 N)^{N+2}}{4\, c_0\, C(N)} < \frac{1}{2\, b_0}$$

if $C(N)$ is chosen so that $C(N) > b_0\, c_0^{-1} C_2\, 2^{N-1} N(1+C_1 N)^{N+2}$. Therefore, we obtain (4.3) for every $z \in B \subset S \cap \Delta \cap D_j$.

Step 3. *A lower bound for the Jacobian.*

We begin with an integral representation of the Jacobian. Assume that U is a convex open set. (We will take $U = b + B$ later.) Let $u, v, w \in U$. Let θ be the largest of the interior angles of the triangle uvw. Then $\pi/3 \le \theta \le \pi$. By renaming the points if necessary, we may assume that the angle at v equals θ and that $|v-u| \le |w-v|$. We have the representation

$$J(u, v, w) = \int_u^v \int_v^w \int_{s_1}^{s_2} \phi'''(z)\, dz\, ds_2\, ds_1 \tag{4.6}$$

where each integral is regarded as a line integral over a line segment. (This is where we need the convexity of U.)

By factoring out a unit complex number, we may also assume that $v-u$ is a positive real number. This amounts to having the vector $\overrightarrow{uv}$ horizontal and pointing to the right. We parametrize the line integrals above by setting $s_1 = u + (v-u)t_1$, $s_2 = v + (w-v)t_2$, and $z = s_1 + (s_2 - s_1)t_3$, with $0 \le t_j \le 1$, to obtain

$$J(u, v, w) = (v-u)(w-v) \int_0^1 \int_0^1 \int_0^1 [s_2(t_2) - s_1(t_1)]\phi'''(z(t_1, t_2, t_3))\, dt_3\, dt_2\, dt_1.$$

Let us now put $s_2 - s_1 = s_2(t_2) - s_1(t_1) = \alpha + i\beta$ and $H_j \cdot z^j = a + i\delta$, where $H_j = \prod_{\ell=j+1}^N |z_\ell|$. Thus, we have

$$\phi'''(z) = \pm(a + i\delta)(\xi + i\eta).$$

By our assumptions, β is single-signed. Let us assume $\beta \ge 0$ for the sake of definiteness. Since $|\delta| \le c\,\varepsilon a$ when $z \in B \subset \Delta$, we have

$$\begin{aligned} \operatorname{Re}[(s_2 - s_1)\phi'''(z)] &= (\alpha a - \beta\delta)\xi - (\beta a + \alpha\delta)\eta \\ &= \alpha a\xi - \beta a\eta + O(\varepsilon|s_2 - s_1|\, a\,\xi); \end{aligned} \tag{4.7}$$

$$\begin{aligned} \operatorname{Im}[(s_2 - s_1)\phi'''(z)] &= (\alpha a - \beta\delta)\eta + (\beta a + \alpha\delta)\xi \\ &= \alpha a\eta + \beta a\xi + O(\varepsilon|s_2 - s_1|\, a\,\xi). \end{aligned} \tag{4.8}$$

Note that the signs of (4.7) and (4.8) do not affect our argument, because we estimate the absolute value of the Jacobian $J(u, v, w)$ from below as follows:

$$|J(u, v, w)| \gtrsim |v-u||w-v| \cdot \left| \int_0^1 \int_0^1 \int_0^1 \operatorname{Im}[(s_2 - s_1)\phi'''(z)]\, dt_3 dt_2 dt_1 \right|.$$

Fix a set B as above and assume that $u, v, w \in B \subset (S \cap \Delta \cap E_j)$, with $E_j = G_j$ or D_j.

Let us now consider the following two cases separately: (3.i) $\pi/3 \le \theta < \pi/2$, and (3.ii) $\pi/2 \le \theta \le \pi$. (Recall that θ is the interior angle at the vertex v of the triangle uvw.)

The case (3.i): $\pi/3 \le \theta < \pi/2$. We claim that

$$\int_{\{\beta \ge |\alpha|/2\}} \beta\, a\xi \ge c\, G$$

where we put

$$G = \int_0^1 \int_0^1 \int_0^1 |s_2 - s_1| \cdot H_j \cdot x^j\, \xi\, dt_3 dt_2 dt_1.$$

Recall that $H_j = \prod_{k=j+1}^N |z_k|$ and $z = x + iy$.

This may be seen as follows. Fix $t_1 \in [0, 1]$. Let $t_2(t_1)$ be the smallest value of $t_2 \in [0, 1]$ such that $\beta \ge \alpha/2 > 0$, i.e. $\operatorname{Im}(s_2(t_2) - s_1(t_1)) \ge (1/2)\operatorname{Re}(s_2(t_2) - s_1(t_1)) > 0$ for $t_2 \ge t_2(t_1)$. If $|w-v|$ is much larger than $|v-u|$, then the term $x^j = [\operatorname{Re}(z)]^j$, which is comparable to $\operatorname{Re}[z^j]$ for $z \in \Delta$, may vary a lot in the triangle uvw. Thus, we split the integral into two parts. (This splitting is not necessary when $|w-v| \le 2|v-u|$, say.)

By our assumptions it follows that $1 - t_2(t_1) \ge t_2(t_1)$ for $t_1 \in [0, 1]$. Note that the triangle with vertices at u, v and $s_2(2\, t_2(0))$ is contained in the ball $B(v, 2|v|\varepsilon)$, centered at v. Also, for all $z \in B(v, 2|v|\varepsilon)$, we have $x \approx |v|$. Thus, for $t_1 \in [0, 1]$, we have

$$\begin{aligned} \int_{[t_2(t_1),\,1]} \int_0^1 |s_2 - s_1|\, H_j \cdot x^j\, \xi\, dt_3 dt_2 &\ge \int_{[2t_2(t_1),\,1]} \int_0^1 |s_2 - s_1|\, H_j \cdot x^j\, \xi\, dt_3 dt_2 \\ &\quad + c \int_{[t_2(t_1),\,2t_2(t_1)]} \int_0^1 |s_2 - s_1|\, H_j \cdot |v|^j\, \xi\, dt_3 dt_2 =: J_1 + cJ_2. \end{aligned} \tag{4.9}$$

Given $s_1 = s_1(t_1)$, let $L = L(t_1)$ be the distance from s_1 to the segment vw. Then the lengths of segments $[s_1, s_2]$ with $s_2 = s_2(t_2)$ for any $t_2 \in [0,\ 2t_2(t_1)]$ are all comparable to L. In fact, $L \le |s_1 - s_2| \le 2L$. Also, we have $\xi \approx |g(z)| \approx 1$ on B, where the implied constants depend only on N. These facts imply that

$$J_2 \approx \int_{[0,\, t_2(t_1)]} \int_0^1 |s_2 - s_1|\, H_j \cdot x^j\, \xi\, dt_3 dt_2 =: J_3.$$

Thus, integrating both sides of the inequality (4.9) in $t_1 \in [0, 1]$ gives

$$\int_{\{\beta \ge \alpha/2 > 0\}} \beta a \xi \gtrsim \int_0^1 J_1 + c \int_0^1 J_2 \ge \int_0^1 J_1 + \frac{c}{2} \int_0^1 J_2 + \frac{c}{2} \int_0^1 c_1 J_3 \gtrsim G$$

since $G \approx \int_0^1 (J_1 + J_2 + J_3)\, dt_1$.

Hence, it follows from (4.3) that

$$\begin{aligned}
\int_0^1 \int_0^1 \int_0^1 \operatorname{Im}\,[(s_2 - s_1)\phi'''(z)]\, dt_3\, dt_2\, dt_1 &= \int \beta a \xi + \int \alpha a \eta + O(\varepsilon G) \\
&\ge \int_{\{\beta \ge \alpha/2 > 0\}} \beta a \xi - b_0^{-1} \int |\alpha| a \xi + O(\varepsilon G) \ge c_2\, G - b_0^{-1} C_3 G + O(\varepsilon G) \\
&\ge (c_2 - b_0^{-1} C_3 - C_4 \varepsilon) G \ge \frac{c_2}{2} G
\end{aligned}$$

if b_0 is chosen sufficiently large and ε sufficiently small. Therefore, we may conclude that

$$\begin{aligned}
|J(u, v, w)| &\gtrsim |v - u||w - v| \left| \int_0^1 \int_0^1 \int_0^1 |s_2 - s_1| \cdot H_j \cdot x^j\, dt_3 dt_2 dt_1 \right| \\
&\gtrsim |v - u||w - v| \cdot \left| \int_0^1 \int_0^1 \int_0^1 (s_2 - s_1) \cdot H_j \cdot z^j\, dt_3 dt_2 dt_1 \right| \\
&= H_j \cdot \left| \int_u^v \int_v^w \int_{s_1}^{s_2} z^j\, dz ds_2 ds_1 \right|. \qquad (4.10)
\end{aligned}$$

Here we used the fact that $\xi \approx 1$.

Next, observe that the last integral is precisely (a constant multiple of) the determinant of the Jacobian of the transformation $(u, v, w) \mapsto \Gamma(u) + \Gamma(v) + \Gamma(w)$, when we take $\Gamma(z) = (z, z^2, z^{j+3})$. Therefore, one can use Lemma 3.3 (with $d = 3$, $m = 1$) to show that the last integral is bounded below by a constant multiple of

$$H_j \cdot \mathbf{v}(u, v, w)\ \max\{|u|^j,\ |v|^j,\ |w|^j\}.$$

This is equivalent to

$$\mathbf{v}(u, v, w) \max\{|\phi'''(u)|, |\phi'''(v)|, |\phi'''(w)|\}$$

for $u, v, w \in B$ with $B \subset S \cap \Delta \cap E_j$, when $E_j = G_j$ (by Lemma 4.1) or when $E_j = D_j$ (by the representation (4.2) and the fact that $|z| \approx |z_j|$ on D_j). This yields the desired lower bound (4.1) when $\pi/3 \le \theta < \pi/2$. (Recall that θ is the interior angle at the vertex v of the triangle uvw.)

The case (3.ii): $\pi/2 \le \theta \le \pi$. In this case, we have $\alpha \ge 0$ and $\beta \ge 0$ (or $\beta \le 0$). This case is easier than the previous one, since there is no cancellation in either of the integrals $\int \alpha a \xi$ or $\int \beta a \xi$. Hence, in this case we have $\int \alpha a \xi + |\int \beta a \xi| = \int \alpha a \xi + \int |\beta| a \xi \ge c\, G$. If $\int |\beta| a \xi \ge (c/2) G$, then we get $|\int \operatorname{Im}\,[(s_2 - s_1)\phi'''(z)]| \gtrsim G$, as before. If not, then we have $\int \alpha a \xi \ge (c/2) G$, and so we would get $|\int \operatorname{Re}\,[(s_2 - s_1)\phi'''(z)]| \gtrsim G$, instead. In either case, we obtain (4.10) for $u_i \in b + B \subset b + (S \cap \Delta \cap E_j)$, $1 \le i \le 3$, and the rest of the argument is the same as the previous case (3.i).

Step 4. *Completion of the proof of* Lemma 4.2.

We will finish the argument by stating how to make up the collection $\{B\}$ to cover $\mathbb{C}$. The sets $\{B\}$, which arose from all the decomposition steps above, need to be translated by b, and then one gets $b + S = \cup(b + B)$, except for a null-set. To be precise, each distinct root b of $\phi'''(z)$ contributes its own collection $\{b + B\}$ to cover $b + S$, where $S = S(b)$ depends on b. In fact, $b + S(b) = \{z \in \mathbb{C} :\ |z - b| < |z - b'|,\ \forall b' \ne b\}$, where $\{b'\}$ is the zero set of $\phi'''(z)$. Finally, the collection of all these sets gives the desired decomposition of $\mathbb{C}$, i.e. $\mathbb{C} = \cup_b [b + S(b)] = \cup_b \cup_{B \subset S(b)} (b + B)$, ignoring a null-set. It just remains to rename the sets $b + B$ as B so that $\mathbb{C} = \cup B$, except for a null-set. This completes the proof of Lemma 4.2. □

A sublevel set estimate. We also need the following simple observation on the complex form of the Vandermonde determinant.

Lemma 4.4. *Let* $\mathbf{v}(h) = \mathbf{v}(h_1, h_2, \ldots, h_d) = \prod_{1\le i<j\le d} |h_i - h_j|$. *Then there is a constant* C_d *such that for each fixed* $h_1 \in \mathbb{C}$,

$$|\{h = (h_1, h') = (h_1, h_2, \ldots, h_d) \in \mathbb{C}^d : \mathbf{v}(h) \le u\}| \le C_d\, u^{4/d}, \quad \forall u > 0,$$

where $|\cdot|$ *denotes the* $2(d-1)$*-dimensional Lebesgue measure.*

Proof. Without loss of generality we may assume $h_1 = 0$, by making a translation. Write $x = (x_2, \ldots, x_d)$ and $y = (y_2, \ldots, y_d)$, where $x_j = \mathrm{Re}\,(h_j)$ and $y_j = \mathrm{Im}\,(h_j)$. Then the set $G = \{h' = (h_2, \ldots, h_d) \in \mathbb{C}^{d-1} : |\mathbf{v}(0, h')| \le u\}$ is contained in $\{x \in \mathbb{R}^{d-1} : |\mathbf{v}(0, x)| \le u\} \times \{y \in \mathbb{R}^{d-1} : |\mathbf{v}(0, y)| \le u\}$, since $|\mathbf{v}(0, h')| \ge |\mathbf{v}(0, x)|$ and $|\mathbf{v}(0, h')| \ge |\mathbf{v}(0, y)|$. Thus it follows from the corresponding result in the real case (*cf.* [18,3]) that the measure $|G|$ in $\mathbb{R}^{2(d-1)}$ is bounded by $C_d(u^{2/d})^2$. □

5. Interpolation of multilinear operators with symmetries

The following lemma was proved in [5]. It is a variant of an interpolation theorem for r-convex spaces obtained in [3]. The original version for Banach spaces, sometimes called the 'multilinear trick', goes back to Christ [10].

For a complete quasi-normed space X, let $\ell^p_\alpha(X)$ be the space of the vector-valued sequences $f = \{f_j\}_{j\in\mathbb{Z}}$ such that

$$\|f\|_{\ell^p_\alpha(X)} = \left(\sum_{j\in\mathbb{Z}} [2^{\alpha j}\|f_j\|_X]^p\right)^{1/p} < \infty.$$

Theorem 5.1. *Let* $n \ge 3$ *be an integer and let* $0 < r \le 1$. *Suppose that* $\delta_1, \ldots, \delta_n$ *are real numbers so that the* δ_i *are not all equal for* $i \ge 2$. *Let* V *be an* r*-convex*[3] *Lorentz space, and let* $\overline{X} = (X_0, X_1)$ *be a couple of compatible complete quasi-normed spaces. Let* T *be a multilinear operator defined on* n*-tuples of* $(X_0 + X_1)$*-valued sequences and suppose that for every permutation* π *on* n *letters we have the inequality*

$$\|T(f_{\pi(1)}, \ldots, f_{\pi(n)})\|_V \le \|f_1\|_{\ell^r_{\delta_1}(X_1)} \prod_{i=2}^n \|f_i\|_{\ell^r_{\delta_i}(X_0)}. \tag{5.1}$$

Then there is a constant C *such that*

$$\|T(f_1, \ldots, f_n)\|_V \le C \prod_{i=1}^n \|f_i\|_{\ell^{nr}_\sigma(\overline{X}_{\frac{1}{n}, nr})}, \quad \sigma = \frac{1}{n}\sum_{i=1}^n \delta_i. \tag{5.2}$$

6. Proof of Theorem 1.3

About this proof. We will assume that the conclusion (4.1) of Lemma 4.2 is valid for a given $d \ge 3$, and then formally deduce from this assumption the d-dimensional version of (1.7), which is in the same form as (1.5). Actually, we will prove the dual estimate (1.14). Since Lemma 4.2 has been established for $d = 3$, this shows Theorem 1.3. We decided to present the proof in this way, showing most steps in general dimension $d \ge 3$, since they are needed again in the next section to prove Theorem 1.2 for all $d \ge 3$.

Offspring curves. Write $\gamma(z) = (z, z^2, \ldots, z^{d-1}, \phi(z))$, where $\phi(z) = \sum_{i=0}^N \alpha_i z^i$, $\alpha_i \in \mathbb{C}$. Let us put

$$\Gamma(z) = (P_1(z), \ldots, P_{d-1}(z), \phi(z)) \tag{6.1}$$

where $P_j(z) = z^j +$ *lower order terms*, and $\phi(z) = \sum_{i=0}^N \alpha_i z^i$ with some new coefficients $\alpha_i \in \mathbb{C}$.

The expression $\Gamma(z)$ is an analogue of the 'offspring curves' in the terminology of [15,18]. For instance, if $\Gamma(z)$ is as above with $|\alpha_i| \le 1$ and $|h_j| \le 2$, for $1 \le j \le d$, then the expression $\Gamma_1(z, h) = d^{-1}\sum_{j=1}^d \Gamma(z + h_j)$ is again in the form (6.1), and the coefficients $\widetilde{\alpha}_i$ of the last component $\phi_1(z)$ of $\Gamma_1(z, h)$ satisfy $|\widetilde{\alpha}_i| \le C(d, N)$ for some constant $C(d, N)$. (See (7.4).)

Two crucial lower bounds. As in [5] (see Section 4), the following two lower bounds will play crucial roles here. The first lower bound concerns the (real) *Jacobian* $J_{\mathbb{R}}(z_1, \ldots, z_d)$ of the transformation $(z_1, \ldots, z_d) \mapsto \sum_{j=1}^d \Gamma(z_j)$, considered as a real mapping, while the second one is about the *torsion* $\tau(z, h)$ of the offspring curves given by $z \mapsto \Gamma(z, h) = \sum_{j=1}^d \Gamma(z + h_j)$, for fixed $h = (h_1, \ldots, h_d)$.

(i) *The Jacobian bound*:

$$J_{\mathbb{R}}(z_1, \ldots, z_d) \ge c(d, N)\,\mathbf{v}(z_1, \ldots, z_d)^2 \max_{j=1,\ldots,d} w(z_j)^{\frac{d^2+d}{2}} \tag{6.2}$$

where $\mathbf{v}(z_1, \ldots, z_d) = \prod_{1\le i<j\le d} |z_i - z_j|$.

[3] This means that there is a constant C such that $\|\sum_{j=1}^M f_j\|_V^r \le C\sum_{j=1}^M \|f_j\|_V^r$ for all $M \ge 1$ and $f_j \in V$. It is crucial that C is independent of M. The Lorentz space $L^{r,\infty}$ is known to be r-convex for $0 < r < 1$. (*cf.* [21,26])

(ii) *The torsion bound*:

$$|w(z,h)| = |\tau(z,h)|^{4/(d^2+d)} \geq c(d,N) \max_{j=1,\ldots,d} w(z+h_j) \tag{6.3}$$

for $z_j = z + h_j \in B$, whenever B is one of the sets in Lemma 4.2. Here, $h = (h_1, \ldots, h_d)$ with $h_j \in \mathbb{C}$, $h_1 = 0$, and $w(z)$ is given by (1.3) with $\gamma(z)$ replaced by $\Gamma(z)$.

These are (6.18) and (6.13), respectively. The precise statements can be found there. We emphasize that for our argument to work (more precisely, for the use of Lemma 5.1 to be valid), at least one of these two lower bounds must be in the stronger form where, on the right-hand side of the inequality, instead of the usual *geometric mean* the *arithmetic mean* (or equivalently, the *maximum* as written above) of the relevant terms is used.

The following proof is an adaptation of an argument used already in [5]. It is arranged somewhat differently here, because unlike in [5] we cannot assume that the result is known for the 'nondegenerate' case (see [3]) in this context. Thus, both the nondegenerate and degenerate cases are treated simultaneously here. We give the proof in some detail, for some of the necessary changes may not be obvious. But our presentation will be somewhat sketchy at places. We refer the reader to Sections 4 and 5 of [5] for more details on such points.

Observe that it suffices to consider the case $N \geq d$, since for $0 \leq N < d$, we have $\gamma^{(d)}(z) \equiv 0$, and so $w(z) \equiv 0$ and there is nothing to prove. By a scaling argument it suffices to prove the estimate for functions f supported in a fixed ball, say, $B(0,1)$ in $\mathbb{C}$ or $\mathbb{R}^2$.

Define

$$w(z) = |\det(\Gamma'(z), \Gamma''(z), \ldots, \Gamma^{(d)}(z))|^{4/(d^2+d)}.$$

A calculation shows that

$$w(z)^{\frac{d^2+d}{4}} = c_d\, |\phi^{(d)}(z)| \tag{6.4}$$

where $c_d = 2! \cdots (d-1)!$.

Now, for $\lambda > 1$, define

$$T_\lambda^\Gamma f(x) = \psi(x) \int_{B(1)} e^{i\lambda x\cdot\Gamma(z)} f(z)\, w(z) d\mu(z), \quad x \in \mathbb{R}^{2d}, \tag{6.5}$$

where $\psi(x)$ is a nonnegative cutoff function and $B(r) = B(0,r)$, $r > 0$.

Put $Q = q_d = (d^2+d+2)/2$, and define

$$A_\lambda = \lambda^{2d/Q} \cdot \sup_\Gamma \|T_\lambda^\Gamma\|_{L^Q(B(1),\, w d\mu) \to L^{Q,\infty}(\mathbb{R}^{2d})} \tag{6.6}$$

where the supremum is taken over all offspring curves Γ as in (6.1) with $|\alpha_i| \leq 1$. Recall that Γ is given by (6.1) and $\phi(z) = \sum_{i=0}^N \alpha_i z^i$. (Notice that the cutoff function $\psi(x)$ in (6.5) may be replaced by a translation $\psi(x-x_0)$ without affecting the norm bound, since a factor of the form $e^{i\lambda x_0\cdot\Gamma(z)}$ may be absorbed into the function $f(z)$.)

Let us first see that $A_\lambda < \infty$ for each $\lambda > 1$. By Hölder's inequality and (6.4) we have

$$\begin{aligned}\|w\|_{L^1(B(1),\, d\mu)} &\leq |B(1)|^{\frac{d^2+d-4}{d^2+d}} \cdot \|w^{\frac{d^2+d}{4}}\|_{L^1(B(1),\, d\mu)}^{\frac{4}{d^2+d}} \\ &\leq |B(1)|^{\frac{d^2+d-4}{d^2+d}} \cdot (c_d)^{\frac{4}{d^2+d}} \|\phi^{(d)}\|_{L^1(B(1),\, d\mu)}^{\frac{4}{d^2+d}} \leq C_{d,N}\end{aligned}$$

for some constant $C_{d,N}$ uniform in the coefficients $\alpha = (\alpha_0, \ldots, \alpha_N)$ of $\phi(z)$ with $|\alpha_i| \leq 1$, $0 \leq i \leq N$.

So, by Hölder's inequality we obtain

$$\|f\|_{L^1(B(1),\, wd\mu)} \leq \|w\|_{L^1(B(1),\, d\mu)}^{1/Q'} \|f\|_{L^Q(B(1),\, wd\mu)} \leq C_{d,N}^{1/Q'} \|f\|_{L^Q(B(1),\, wd\mu)}.$$

Since $|T_\lambda^\Gamma f(x)| \leq |\psi(x)| \cdot \|f\|_{L^1(B(1),\, wd\mu)}$, the last inequality implies that

$$\begin{aligned}\|T_\lambda^\Gamma f\|_{L^{Q,\infty}(\mathbb{R}^{2d})} &\leq \|\psi\|_{L^{Q,\infty}(\mathbb{R}^{2d})} \|f\|_{L^1(B(1),\, wd\mu)} \\ &\leq \|\psi\|_{L^{Q,\infty}(\mathbb{R}^{2d})} \cdot C_{d,N}^{1/Q'} \|f\|_{L^Q(B(1),\, wd\mu)}.\end{aligned}$$

Hence, it follows that for each $\lambda > 1$,

$$A_\lambda \leq \lambda^{2d/Q} \cdot C_{d,N}^{1/Q'} \|\psi\|_{L^{Q,\infty}(\mathbb{R}^{2d})} < \infty. \tag{6.7}$$

Our goal is to show that $A_\lambda \leq C(d,N)$, independent of $\lambda > 1$. This, in turn, would imply that

$$\|T_\lambda^\Gamma f\|_{L^{Q,\infty}(\mathbb{R}^{2d})} \leq C(d,N)\, \lambda^{-2d/Q} \|f\|_{L^Q(B(1),\, wd\mu)}, \quad \lambda > 1 \tag{6.8}$$

uniformly in $\alpha = (\alpha_0, \ldots, \alpha_N)$ with $|\alpha_i| \le 1$, $0 \le i \le N$, if f is supported in $B(1)$. Assuming (6.8), it is easy to finish the proof of (1.14). First we take $\Gamma(z) = \gamma(z)$. Then we make a change of variables $x \mapsto \lambda^{-1}x$ to remove the factor $\lambda^{-2d/Q}$, and next we take the limit as $\lambda \to \infty$ to remove the cutoff function $\psi(x)$. Finally, summing over the B's, where the B are as in Lemma 4.2, we obtain (1.14) for f supported in $B(1)$. Then a scaling argument extends (1.14) to functions f supported in $\mathbb{C}$.

It remains to show $A_\lambda \le C(d, N)$, and consequently (6.8). Fix $\lambda > 1$. Also fix $\Gamma(z)$ as in (6.1) with $\alpha = (\alpha_0, \ldots, \alpha_N)$, $\alpha_i \in \mathbb{C}$ and $|\alpha_i| \le 1$, $0 \le i \le N$. Let $|h_j| \le 2$, $1 \le j \le d$. Put $\Gamma(z, h) = \sum_{j=1}^{d} \Gamma(z + h_j)$. Then $\Gamma(z, h)$ is in the form

$$\Gamma(z, h) = (d \cdot P_1(z), \ldots, d \cdot P_{d-1}(z), \phi_1(z)) \tag{6.9}$$

where the $P_j(z)$ are as in (6.1), with the leading coefficient 1, but some new coefficients for the lower order terms, and $\phi_1(z) = \sum_{i=0}^{N} \tilde{a}_i z^i$ with $|\tilde{\alpha}_i| \le c_* = C(d, N)$. The constants $d, \ldots, d, c_*$ and c_* can be factored from $\Gamma(z, h)$ and incorporated into x. Namely, we may rewrite

$$x \cdot \Gamma(z, h) = y \cdot \Gamma_1(z, h).$$

Here, $y = (d x_1, \ldots, d x_{d-2}, c_* x_{d-1}, c_* x_d) = xL$, where L is a $d \times d$ diagonal matrix, and

$$\Gamma_1(z, h) = (P_1(z), \ldots, P_{d-1}(z), c_*^{-1} \phi_2(z))$$

is an offspring curve as in (6.1), of which the last component has coefficients $\widetilde{\alpha_i}$ with $|\widetilde{\alpha_i}| \le 1$. The change of variables $x \mapsto y$ changes the cutoff function to

$$\psi(yL^{-1}) = \psi\left(\frac{y_1}{d}, \ldots, \frac{y_{d-2}}{d}, \frac{y_{d-1}}{c_*}, \frac{y_d}{c_*}\right).$$

Since $\psi(yL^{-1})$ is bounded by the sum of no more than $C(d, N)$ translates of $\psi(y)$, we may apply the definition of A_λ. This only increases the constant by a factor $C(d, N)$.

By writing $B(1) = B(0, 1)$ as a union of the sets $B(1) \cap B$, where the B are as in Lemma 4.2, we may assume that f is supported in B. We may also assume that $B \subset B(1)$. (Otherwise, replace B with $B(1) \cap B$.) Thus, we may rewrite

$$T_\lambda^\Gamma f(x) = \psi(x) \int_B e^{i\lambda x \cdot \Gamma(z)} f(z)\, w(z)\, d\mu(z), \quad x \in \mathbb{R}^{2d}. \tag{6.10}$$

Let us put

$$\begin{aligned} M_\lambda(f_1, \ldots, f_d)(x) &= \prod_{j=1}^{d} (T_\lambda^\Gamma f_j)(x) = \psi(x)^d \int_{B^d} e^{i\lambda x \cdot \sum_{i=1}^{d} \Gamma(z_j)} \prod_{j=1}^{d} [(f_j\, w)(z_j)]\, d\mu(z_1) \cdots d\mu(z_d) \\ &= \psi(x)^d \int_{B(2)^{d-1}} \int_{B_h} e^{i\lambda x \cdot \Gamma(z,h)} \prod_{j=1}^{d} [(f_j\, w)(z + h_j)]\, d\mu(z)\, d\mu(h_2) \cdots d\mu(h_d). \end{aligned}$$

Here, B_h is the intersection of the sets $B - h_j$ (translates of B) over the indices $j = 1, \ldots, d$.

Next, as in [2] we define the decomposed operators

$$M_{\lambda,k}(f_1, \ldots, f_d)(x) = \psi(x)^d \int_{S_k} \int_{B_h} e^{i\lambda x \cdot \Gamma(z,h)} \prod_{j=1}^{d} [(f_j\, w)(z + h_j)]\, d\mu(z)\, d\mu(h_2) \cdots d\mu(h_d) \tag{6.11}$$

where

$$S_k = \{h' = (h_2, \ldots, h_d) \in B(2)^{d-1} : 2^{-k-1} < \mathbf{v}(0, h') \le 2^{-k}\}$$

for $k \in \mathbb{Z}$. Recall that $\mathbf{v}(h) = \prod_{1 \le i < j \le d} |h_i - h_j|$.

An estimate at $q = Q$. By the considerations about $\Gamma(z, h)$ given in the paragraph containing (7.4) and from the definition (6.6) of A_λ, it follows that

$$\left\| \psi(x) \int_{B_h} e^{i\lambda x \cdot \Gamma(z,h)} f(z) \cdot w(z, h)\, d\mu(z) \right\|_{Q,\infty} \lesssim \lambda^{-\frac{2d}{Q}} A_\lambda \|f\|_{L^Q(B_h,\, w(z,h) d\mu)} \tag{6.12}$$

uniformly in h. Here, $w(z, h) = |\tau(z, h)|^{4/(d^2+d)}$, and

$$\tau(z, h) = \det(\Gamma'(z, h), \ldots, \Gamma^{(d)}(z, h)).$$

We have

$$|\tau(z, h)| = d^{d-1} 2! \cdots (d-1)! \cdot \left| \sum_{j=1}^{d} \phi^{(d)}(z + h_j) \right|.$$

Thus, as in the proof of Lemma 4.2 (or Lemma 3.3) we obtain

$$|\tau(z,h)| \geq c_{d,N} \sum_{i=1}^{d} \left|\phi^{(d)}(z+h_i)\right| \geq c_{d,N} \max_{1\leq i\leq d} w(z+h_i)^{(d^2+d)/4} \tag{6.13}$$

for $z+h_i \in B$. Here we used (6.4). Now set

$$w_*(z,h) := \prod_{i=1}^{d} w(z+h_i)^{a_i}$$

for some constants $a_i \in [0,1]$ with $\sum_{i=1}^{d} a_i = 1$. We will choose a_i suitably later so that the condition $\delta_2 \neq \delta_3$ (the δ_j are to be defined later) is satisfied, which will allow us to apply the interpolation theorem, Theorem 5.1. (See the paragraph containing (6.23).) Thus, as was mentioned in the first paragraph of this section, the fact that we have an arithmetic mean instead of a geometric mean as the lower bound in (6.13) plays a key role in our argument.

The inequality (6.13) implies that

$$w(z,h) \geq c\, w_*(z,h) \tag{6.14}$$

for $z \in B_h$, where $c = c(d,N) > 0$ is a constant independent of z, h. (Recall that B_h is the intersection of the sets $-h_j + B$, $j = 1, \dots, d$.)

If we write $w_*(z,h) = g(z,h)w(z,h)$ for some nonnegative function $g(z,h) \leq c$, then we may apply (6.12) with $f(z,h)$ replaced by $f(z,h)g(z,h)$. Since $g(z,h)^Q \leq Cg(z,h)$, this gives

$$\left\| \psi(x) \int_{B_h} e^{i\lambda x\cdot\Gamma(z,h)} f(z)\cdot w_*(z,h)\, d\mu(z) \right\|_{Q,\infty} \leq C\lambda^{-\frac{2d}{Q}} A_\lambda \|f\|_{L^Q(w_*(z,h)d\mu)}.$$

(See Observation 5.1 in [5] for more details about this argument.)

It follows then from an analogue of Minkowski's inequality, by using an equivalent 'norm' on $L^{Q,\infty}$ for this purpose (see Section 4 of [5]), that

$$\begin{aligned}
\|M_{\lambda,k}(f_1,f_2,\dots,f_d)\|_{Q,\infty} &\leq C \int_{S_k} \left\| \psi(x) \int_{B_h} e^{i\lambda x\cdot\Gamma(t,h)} \prod_{j=1}^{d} [f_j(z+h_j)w(z+h_j)^{1-a_j}]\cdot w_*(z,h)\, d\mu(z) \right\|_{Q,\infty} \\
&\quad \times d\mu(h_2)\cdots d\mu(h_d) \\
&\leq C\,\lambda^{-\frac{2d}{Q}} A_\lambda \int_{S_k} \left\| \prod_{j=1}^{d} [f_j(z+h_j)w(z+h_j)^{1-a_j}] \right\|_{L^Q(w_*(z,h)d\mu)} d\mu(h_2)\cdots d\mu(h_d) \\
&= C\,\lambda^{-\frac{2d}{Q}} A_\lambda \int_{S_k} \left\| \prod_{j=1}^{d} \left[f_j(z+h_j) w(z+h_j)^{1-\frac{a_j}{Q'}} \right] \right\|_{L^Q(d\mu)} d\mu(h_2)\cdots d\mu(h_d).
\end{aligned}$$

We will now apply Hölder's inequality to bound the inner norm and also use the sublevel set estimate in Lemma 4.4 with $u = 2^{-k}$. This gives

$$\|M_{\lambda,k}(f_1,\dots,f_d)\|_{Q,\infty} \leq C\,\lambda^{-\frac{2d}{Q}} A_\lambda \cdot 2^{-\frac{4k}{d}} \prod_{j=1}^{d} \left\| f_j w^{1-\frac{a_j}{Q'}} \right\|_{L^{q_j}(d\mu)} \tag{6.15}$$

where $\sum_{j=1}^{d} 1/q_j = 1/Q$ for some numbers q_j, $1 \leq q_j \leq \infty$, to be chosen later.

Let us now put

$$\Omega_i = \{z \in \mathbb{C} :\ 2^{i-1} \leq w(z) < 2^i\}, \quad i \in \mathbb{Z}.$$

The triangle inequality implies that

$$\|f\, w^\alpha\|_{L^p(d\mu)} = \left\| \sum_{i\in\mathbb{Z}} \chi_{\Omega_i} f\, w^\alpha \right\|_{L^p(d\mu)} \leq C \sum_{i\in\mathbb{Z}} 2^{i\alpha} \|f\chi_{\Omega_i}\|_{L^p(d\mu)}$$

for $\alpha \in \mathbb{R}$. Hence, it follows that

$$\|M_{\lambda,k}(f_1,\dots,f_d)\|_{Q,\infty} \leq C\lambda^{-\frac{2d}{Q}} A_\lambda \cdot 2^{-\frac{4k}{d}} \prod_{j=1}^{d} \|f_j\|_{\ell^1_{\alpha_j}(L^{q_j}(d\mu))} \tag{6.16}$$

where we put $\alpha_j = 1 - a_j/Q'$. Here the expression $\|f\|_{\ell^p_\alpha(X)}$ stands for

$$\|\{f\chi_{\Omega_i}\}\|_{\ell^p_\alpha(X)} = \left(\sum_{i\in\mathbb{Z}} \left[2^{\alpha i} \|f\chi_{\Omega_i}\|_X\right]^p \right)^{1/p}$$

where X is a Banach space (or a complete quasi-normed space) of functions on $\mathbb{R}^2$. Thus, we identify f with the sequence $\{f\chi_{\Omega_i}\}_{i\in\mathbb{Z}}$.

An L^2 estimate. Next, it follows from Bézout's theorem that the transformation $(z, h_2, \ldots, h_d) \mapsto \Gamma(z, h_2, \ldots, h_d)$ has bounded generic multiplicity $\leq N \cdot (d-1)!$. By Proposition 1.4.10 on p. 51 in [22], the Jacobian of this transformation as a real mapping is given by

$$J_{\mathbb{R}}(z_1, \ldots, z_d) = |J_{\mathbb{C}}(z_1, \ldots, z_d)|^2 = |\det(\Gamma'(z+h_1), \ldots, \Gamma'(z+h_d))|^2$$

for $z_j = z + h_j \in B$, with $h_1 = 0$. Here, $J_{\mathbb{C}}(z_1, \ldots, z_d)$ denotes the determinant of the holomorphic Jacobian matrix for the transformation $(z_1, \ldots, z_d) \mapsto \mathbf{\Gamma}(z_1, \ldots, z_d) = \sum_{j=1}^d \Gamma(z_j) = \sum_{j=1}^d \Gamma(z+h_j)$.

For instance, when $d = 3$, we have

$$J_{\mathbb{C}}(z_1, z_2, z_3) = \int_{z_1}^{z_2}\int_{z_2}^{z_3}\int_{s_1}^{s_2} \phi'''(z)\, dz\, ds_2\, ds_1 \tag{6.17}$$

where each integral is a line integral over a line segment. (In higher dimensions there is a similar representation, defined recursively, which involves integrals of $\phi^{(d)}(z)$. See [4,14,11].)

Hence, by our assumption that Lemma 4.2 holds for $d \geq 3$, it follows that

$$J_{\mathbb{R}}(z_1, \ldots, z_d) \gtrsim \mathbf{v}(h)^2 \cdot \frac{1}{d}\sum_{j=1}^d w(z+h_j)^{\frac{d^2+d}{2}} \geq \mathbf{v}(h)^2 \prod_{j=1}^d w(z+h_j)^{\frac{d^2+d}{2d}} \tag{6.18}$$

if $z_j = z + h_j \in B$. Here, $\mathbf{v}(h) = \prod_{1\leq i<j\leq d} |h_i - h_j|$ with $h_1 = 0$. (See also Remark 4.3.) Here the implied constant $c = c_{d,N} > 0$ depends only on d and N.

Next, we change variables in the integral (6.11) and use the Plancherel theorem. Then we reverse the change of variables and use (6.18) and the sublevel set estimate in Lemma 4.4 to obtain

$$\begin{aligned}
\|M_{\lambda,k}(f_1, \ldots, f_d)\|_2 &\leq C\lambda^{-d}\left(\int_{S_k}\int \prod_{j=1}^d |(f_j\, w)(z+h_j)|^2 J_{\mathbb{R}}(z,h)^{-1} d\mu(z) d\mu(h_2)\cdots d\mu(h_d)\right)^{1/2} \\
&\leq C\lambda^{-d}\left(\int_{S_k}\int \prod_{j=1}^d |(f_j\, w^a)(z+h_j)|^2\, \mathbf{v}(h)^{-2} d\mu(z) d\mu(h_2)\cdots d\mu(h_d)\right)^{1/2} \\
&\leq C\lambda^{-d}\, 2^k 2^{-\frac{2k}{d}} \|f_1\, w^a\|_{L^2(d\mu)} \prod_{j=2}^d \|f_j\, w^a\|_{L^\infty(d\mu)}
\end{aligned}$$

for $a = (3-d)/4$.

By permuting the variables and interpolating the resulting estimates one gets

$$\|M_{\lambda,k}(f_1, \ldots, f_d)\|_2 \leq C\lambda^{-d}\, 2^{\frac{k(d-2)}{d}} \prod_{j=1}^d \|f_j\, w^a\|_{L^{r_j}(d\mu)}$$

for some numbers $1 \leq r_j \leq \infty$, to be chosen later, such that $\sum_{j=1}^d r_j^{-1} = 2^{-1}$. Using the triangle inequality on each norm again gives

$$\|M_{\lambda,k}(f_1, \ldots, f_d)\|_2 \leq C\lambda^{-d}\, 2^{\frac{k(d-2)}{d}} \prod_{j=1}^d \|f_j\|_{\ell_a^1(L^{r_j}(d\mu))}. \tag{6.19}$$

Summation of the estimates. By estimating the distribution function of the sum of $M_{\lambda,k}(f_1, \ldots, f_d)(x)$ over k, using (6.16) and (6.19), we obtain the estimate

$$\begin{aligned}
\left|\left\{\left|\sum_{k=-\infty}^{\infty} M_{\lambda,k}\right| > 2\alpha\right\}\right| &\leq \left|\left\{\left|\sum_{2^k>\beta} M_{\lambda,k}\right| > \alpha\right\}\right| + \left|\left\{\left|\sum_{2^k\leq\beta} M_{\lambda,k}\right| > \alpha\right\}\right| \\
&\leq \lambda^{-2d}\left(\frac{CA_\lambda}{\alpha}\right)^Q \beta^{-\frac{4Q}{d}} \prod_{j=1}^d \|f_j\|_{\ell_{\alpha_j}^1(L^{q_j})}^Q + \lambda^{-2d}\frac{C^2}{\alpha^2}\beta^{\frac{2(d-2)}{d}} \prod_{j=1}^d \|f_j\|_{\ell_a^1(L^{r_j})}^2
\end{aligned}$$

for $\beta > 0$. Choosing the value

$$\beta = \left(\alpha^{2-Q} A_\lambda^Q \prod_{j=1}^d \left[\|f_j\|_{\ell_{\alpha_j}^1(L^{q_j}(d\mu))}^Q \|f_j\|_{\ell_a^1(L^{r_j}(d\mu))}^{-2}\right]\right)^{\frac{d}{2(d-2+2Q)}}$$

yields that

$$\|M_\lambda(f_1,\dots,f_d)\|_{Q/d,\infty} \le C\,\lambda^{-\frac{2d^2}{Q}}\,A_\lambda^{\frac{d-2}{d+2}} \prod_{j=1}^{d} \|f_j\|_{\ell^1_{\alpha_j}(L^{q_j}(d\mu))}^{\frac{d-2}{d+2}} \|f_j\|_{\ell^1_a(L^{r_j}(d\mu))}^{\frac{4}{d+2}}.$$

Here we used the fact that $d-2+2Q = d(d+2)$ and $Q = (d^2+d+2)/2$.

By Lemma A.3 in [5], this implies that

$$\|M_\lambda(f_1,\dots,f_d)\|_{Q/d,\infty} \le C\,\lambda^{-\frac{2d^2}{Q}}\,A_\lambda^{\frac{d-2}{d+2}} \prod_{j=1}^{d} \|f_j\|_{\left(\ell^1_{\alpha_j}(L^{q_j}(d\mu)),\ \ell^1_a(L^{r_j}(d\mu))\right)_{\frac{4}{d+2},1}}.$$

From Lemma A.4 in [5], we have

$$\left(\ell^1_{\alpha_j}(L^{q_j}(d\mu)),\ \ell^1_a(L^{r_j}(d\mu))\right)_{\frac{4}{d+2},1} = \ell^1_{\beta_j}\left(L^{p_j,1}(d\mu)\right)$$

where

$$\frac{1}{p_j} = \frac{d-2}{d+2}\,\frac{1}{q_j} + \frac{4}{d+2}\,\frac{1}{r_j} \quad \text{and} \quad \beta_j = \frac{d-2}{d+2}\,\alpha_j + \frac{4}{d+2}\,a.$$

Thus, we obtain

$$\left\|\prod_{j=1}^{d} T_\lambda^\Gamma f_j\right\|_{Q/d,\infty} \le C\lambda^{-\frac{2d^2}{Q}}\,A_\lambda^{\frac{d-2}{d+2}}\,\|f_1\|_{\ell^1_{\beta_1}(L^{p_1,1}(d\mu))}\|f_2\|_{\ell^1_{\beta_2}(L^{p_2,1}(d\mu))} \prod_{j=3}^{d} \|f_j\|_{\ell^1_{\beta_j}(L^{p_j,1}(d\mu))}. \tag{6.20}$$

On the other hand we can get an alternative estimate by taking $q_j = dQ$ and $\alpha_j = 1 - 1/(dQ')$ for all j in (6.16), and also taking all $r_j = 2d$ in (6.19). Then taking all $f_j = f$ gives

$$\left\|T_\lambda^\Gamma f\right\|_{Q,\infty} \le C\lambda^{-\frac{2d}{Q}}\,A_\lambda^{\frac{d-2}{d(d+2)}}\,\|f\|_{\ell^1_{\delta_0}(L^{Q,1}(d\mu))} \tag{6.21}$$

where $\delta_0 = 1/Q$.

Preparation for interpolation. We will now consider the n-linear symmetric operator $\prod_{j=1}^{n} T_\lambda^\Gamma f_j$ with some $n > Q$. Then we need to estimate its $L^{r,\infty}$ quasi-norm with $r = Q/n < 1$. This is to take advantage of the r-convexity of this space. (See Section 5 and the footnote 3 there.) For simplicity of notation, let us take $n = dQ$. By applying a variant of Hölder's inequality (*cf.* (2.1) in [3]), using (6.20) for the first d factors and (6.21) for the rest, we get

$$\left\|\prod_{j=1}^{dQ} T_\lambda^\Gamma f_j\right\|_{1/d,\infty} \le C(dQ)^d\,\lambda^{-2d^2}\,A_\lambda^{Q\,\frac{d-2}{d+2}}\,\|f_1\|_{\ell^1_{\beta_1}(L^{p_1,1})}\|f_2\|_{\ell^1_{\beta_2}(L^{p_2,1})} \prod_{j=3}^{d} \|f_j\|_{\ell^1_{\beta_j}(L^{p_j,1})} \prod_{j=d+1}^{dQ} \|f_j\|_{\ell^1_{\delta_0}(L^{Q,1})}.$$

Now we may choose $q_1,\dots,q_d$, and $r_1,\dots,r_d$ (hence also $p_1,\dots,p_d$) such that $p_1 \neq p_2$, with p_2 strictly between p_3 and $Q = (d^2+d+2)/2$, and also that $p_3 = \cdots = p_d$ and

$$\frac{1}{p_2} = \frac{d-2}{dQ-2}\,\frac{1}{p_3} + \frac{d(Q-1)}{dQ-2}\,\frac{1}{Q}. \tag{6.22}$$

Note that we have then also

$$\frac{1}{d}\left(\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_d}\right) = \frac{1}{Q}.$$

(In fact, we may choose q_j and r_j such that $1/p_3 = 1/Q - \varepsilon$ for some small $\varepsilon \neq 0$. Also take $1/p_2 = 1/Q - (d-2)\varepsilon/(dQ-2)$ and $1/p_1 = 1/Q + (dQ-1)(d-2)\varepsilon/(dQ-2)$. These choices satisfy the requirements listed above.)

Put $r = 1/d$ and bound each quasi-norm above of the form $\|\cdot\|_{\ell^1_\rho(L^{p,1})}$ by the quasi-norm $\|\cdot\|_{\ell^r_\rho(L^{p,r})}$. With f_1, f_2 fixed, let us permute the remaining functions and take generalized geometric means of the resulting estimates to get

$$\left\|\prod_{j=1}^{dQ} T_\lambda^\Gamma\ f_j\right\|_{1/d,\infty} \le C\lambda^{-2d^2}\,A_\lambda^{Q\,\frac{d-2}{d+2}}\,\|f_1\|_{\ell^r_{\beta_1}(L^{p_1,r})}\|f_2\|_{\ell^r_{\beta_2}(L^{p_2,r})} \prod_{j=3}^{dQ} \|f_j\|_{\ell^r_{\beta_j}(L^{p_j,r})}^{\frac{d-2}{dQ-2}}\ \|f_j\|_{\ell^r_{\delta_0}(L^{Q,r})}^{\frac{d(Q-1)}{dQ-2}}.$$

By (6.22), Lemmas A.3 and A.4 in [5], we obtain

$$\left\|\prod_{j=1}^{dQ} T_\lambda^\Gamma f_j\right\|_{1/d,\infty} \le C\lambda^{-2d^2}\,A_\lambda^{Q\,\frac{d-2}{d+2}}\,\|f_1\|_{\ell^r_{\delta_1}(L^{p_1,r})}\|f_2\|_{\ell^r_{\delta_2}(L^{p_2,r})} \prod_{j=3}^{dQ} \|f_j\|_{\ell^r_{\delta_j}(L^{p_2,r})}$$

where $\delta_1 = \beta_1$, $\delta_2 = \beta_2$ and

$$\delta_j = \frac{d-2}{dQ-2}\beta_j + \frac{d(Q-1)}{dQ-2}\delta_0, \quad 3 \le j \le d.$$

We may choose $a_j \in [0,1]$ such that $\sum_{j=1}^d a_j = 1$, and $\delta_2 \neq \delta_3$. (Recall that $\beta_j = [(d-2)/(d+2)]\alpha_j + [4/(d+2)]a$, $\alpha_j = 1 - a_j/Q'$ and $a = (3-d)/4$. Thus, it is easy to see that we can satisfy the condition $\delta_2 \neq \delta_3$, by choosing a_2 and a_3 suitably.)

Application of the interpolation theorem. We are now in a position to apply Theorem 5.1. Let us take $X_0 = L^{p_2,r}(d\mu)$ and $X_1 = L^{p_1,r}(d\mu)$. It follows from (5.2) with $n = dQ$ and $V = L^{r,\infty}$ for $r = 1/d$ that

$$\left\| \prod_{j=1}^{dQ} T_\lambda^\Gamma f_j \right\|_{1/d,\infty} \le C\lambda^{-2d^2} A_\lambda^{Q\frac{d-2}{d+2}} \prod_{j=1}^{dQ} \|f_j\|_{\ell_s^Q(\overline{X}_{\frac{1}{n},Q})}$$

where $s = [\delta_1 + \delta_2 + (n-2)\delta_3]/n$. Taking all $f_j = f$ yields

$$\|T_\lambda^\Gamma f\|_{Q,\infty} \le C\lambda^{-2d/Q} A_\lambda^{\frac{d-2}{d(d+2)}} \|f\|_{\ell_s^Q(\overline{X}_{\frac{1}{n},Q})}. \tag{6.23}$$

Note that we have $s = 1/Q = 2/(d^2+d+2)$, since

$$\begin{aligned} dQs &= \sum_{j=1}^{dQ} \delta_j = \delta_1 + \delta_2 + (dQ-2)\left(\frac{d-2}{dQ-2}\beta_3 + \frac{d(Q-1)}{dQ-2}\frac{1}{Q}\right) \\ &= \frac{d-2}{d+2}\sum_{j=1}^d \alpha_j + \frac{d(3-d)}{d+2} + \frac{d(Q-1)}{Q} \\ &= \frac{d-2}{d+2}\left(d - \frac{1}{Q'}\right) + \frac{d(3-d)}{d+2} + \frac{d}{Q'} = d. \end{aligned}$$

Moreover, we have

$$\overline{X}_{\frac{1}{n},Q} = (X_0, X_1)_{\frac{1}{n},Q} = (L^{p_2,r}, L^{p_1,r})_{\frac{1}{n},Q} = L^{p,Q} = L^Q(d\mu)$$

since $p_1 \neq p_2$ and

$$\frac{1}{p} := \frac{1}{n}\frac{1}{p_1} + \frac{n-1}{n}\frac{1}{p_2} = \frac{1}{dQ}\left(\frac{1}{p_1} + \frac{1}{p_2} + \frac{dQ-2}{p_2}\right) = \frac{1}{Q}$$

by the choice of $p_1, \ldots, p_d$ made above in the paragraph containing (6.22). Here we also used the fact (*cf.* Theorem 5.3.1 in [7]) that if $p_0 \neq p_1$, then

$$(L^{p_0,r_0}, L^{p_1,r_1})_{\theta,s} = L^{p,s}$$

for $1/p = (1-\theta)/p_0 + \theta/p_1$, $0 < \theta < 1$, and $0 < s \le \infty$. (As usual, $p_j, r_j \in (0,\infty]$, and we assume $r_j = \infty$ when $p_j = \infty$.)

This shows that we have

$$\begin{aligned} \|f\|_{\ell_s^Q(\overline{X}_{\frac{1}{n},Q})} &= \|\{f\chi_{\Omega_k}\}\|_{\ell_{1/Q}^Q(L^Q(d\mu))} \\ &= \left(\sum_{k\in\mathbb{Z}} \left[2^{k/Q}\|f\chi_{\Omega_k}\|_{L^Q(d\mu)}\right]^Q\right)^{1/Q} \approx \|f\|_{L^Q(w\,d\mu)} \end{aligned}$$

where the last equivalence is a consequence of the fact that $w(z) \approx 2^k$ for $z \in \Omega_k$. So, (6.23) implies that

$$\|T_\lambda^\Gamma f\|_{Q,\infty} \le C_{d,N}\lambda^{-\frac{2d}{Q}} A_\lambda^{\frac{d-2}{d(d+2)}} \|f\|_{L^Q(w\,d\mu)}$$

with a constant $C_{d,N}$ independent of $\lambda > 1$ and Γ with $|\alpha_i| \le 1$.

Hence, by the definition (6.6) of A_λ, we obtain

$$A_\lambda \le C_{d,N} A_\lambda^{\frac{d-2}{d^2+2d}}.$$

Since we have $A_\lambda < \infty$ for $\lambda > 1$ by (6.7), it follows that $A_\lambda \le C(d,N) = (C_{d,N})^{(d^2+2d)/(d^2+d+2)}$, for all $\lambda > 1$. Therefore, we may conclude that the estimate

$$\|T_\lambda^\Gamma f\|_{Q,\infty} \le C(d,N)\lambda^{-\frac{2d}{Q}} \|f\|_{L^Q(w\,d\mu)}$$

holds for $Q = (d^2+d+2)/2$, uniformly in $\lambda > 1$ and Γ. This completes the proof of (6.8). Finally, we take $C(N) = \sum_{d=1}^N C(d,N)$. Taking $d = 3$ gives the dual estimate of (1.7). □

7. Proof of Theorem 1.2

The proof in the previous section carries over here with minor modifications. Thus, we only need to indicate how to modify the argument to work in this situation. Here we define offspring curves of $\gamma(z)$ by

$$\Gamma_b(z) = \frac{1}{m}\sum_{i=1}^{m}\gamma(z+b_i)$$

where $b_i \in \mathbb{C}$ and $b_1 = 0$. (We would like to point out that a separate argument is needed here, because we have to consider the offspring curves of the 'monomial curves', which are no longer monomial curves. That is why we proved the Jacobian bound in Lemma 3.3 for the offspring curves of the above form for all $m \geq 1$.)

Again, by a scaling argument it suffices to prove the estimate for functions f supported in $B(0,1)$ in $\mathbb{C}$ or $\mathbb{R}^2$. We only need to divide $B(0,1)$ into a bounded number of narrow sectors with vertex at the origin. By rotation (which is possible by the homogeneity of $\phi(z) = z^N$ as in Section 3), it is enough to show the estimate for f supported in $\Delta = \{z = x+iy \in B(0,1) : 0 < y < \varepsilon x\}$ with some small $\varepsilon = \varepsilon(d,N) > 0$.

Define

$$T_\lambda^{\Gamma_b} f(x) = \psi(x)\int_{\Delta_b} e^{i\lambda x\cdot\Gamma_b(z)} f(z)\, w_b(z) d\mu(z), \quad x \in \mathbb{R}^{2d}, \tag{7.1}$$

where $\psi(x)$ is a nonnegative cutoff function and $\Delta_b = \bigcap_{i=1}^{m}(\Delta - b_i) \subset \Delta$. (Here, $\Delta - a = \{z - a : z \in \Delta\}$ denotes a translation of Δ.)

Recall that $Q = q_d = (d^2+d+2)/2$. Define

$$A_\lambda = \lambda^{2d/Q}\cdot \sup_{\Gamma_b} \|T_\lambda^{\Gamma_b}\|_{L^Q(\Delta_b,\, w_b d\mu)\to L^{Q,\infty}(\mathbb{R}^{2d})} \tag{7.2}$$

where the supremum is taken over all Γ_b, with $b = (b_1,\ldots,b_m) \in \mathbb{C}^m$, $m \in \mathbb{N}$, $b_1 = 0$, and $|b_j| \leq 1$, for $1 \leq i \leq m$. (Note that Δ_b is empty, if $|b_j| > 1$ for some i.)

Let us show that $A_\lambda < \infty$, for each $\lambda > 1$. By Hölder's inequality and (6.4) we have

$$\begin{aligned}\|w_b\|_{L^1(\Delta_b,\, d\mu)} &\leq |\Delta_b|^{\frac{d^2+d-4}{d^2+d}}\cdot \|w_b^{\frac{d^2+d}{4}}\|_{L^1(\Delta_b,\, d\mu)}^{\frac{4}{d^2+d}}\\ &\leq |\Delta|^{\frac{d^2+d-4}{d^2+d}}\cdot\left(m^{-1}\sum_{j=1}^{m}\|\phi^{(d)}(\cdot+b_j)\|_{L^1(\Delta-b_j,\, d\mu)}\right)^{\frac{4}{d^2+d}}\\ &\leq |\Delta|^{\frac{d^2+d-4}{d^2+d}}\cdot\|\phi^{(d)}\|_{L^1(\Delta,\, d\mu)}^{\frac{4}{d^2+d}} \leq C_{d,N}\end{aligned}$$

for some constant $C_{d,N}$ independent of $m \geq 1$ and b. So, by Hölder's inequality we obtain

$$\|f\|_{L^1(\Delta_b,\, w_b d\mu)} \leq \|w_b\|_{L^1(\Delta_b,\, d\mu)}^{1/Q'}\|f\|_{L^Q(\Delta_b,\, w_b d\mu)} \leq C_{d,N}^{1/Q'}\|f\|_{L^Q(\Delta_b,\, w_b d\mu)}.$$

Since $|T_\lambda^{\Gamma_b} f(x)| \leq |\psi(x)|\cdot\|f\|_{L^1(\Delta_b,\, w_b d\mu)}$, the last inequality implies that

$$\begin{aligned}\|T_\lambda^{\Gamma_b} f\|_{L^{Q,\infty}(\mathbb{R}^{2d})} &\leq \|\psi\|_{L^{Q,\infty}(\mathbb{R}^{2d})}\|f\|_{L^1(\Delta_b,\, w_b d\mu)}\\ &\leq \|\psi\|_{L^{Q,\infty}(\mathbb{R}^{2d})}\cdot C_{d,N}^{1/Q'}\|f\|_{L^Q(\Delta_b,\, w_b d\mu)}.\end{aligned}$$

Hence, it follows that for each $\lambda > 1$,

$$A_\lambda \leq \lambda^{2d/Q}\cdot C_{d,N}^{1/Q'}\|\psi\|_{L^{Q,\infty}(\mathbb{R}^{2d})} < \infty. \tag{7.3}$$

It remains to show $A_\lambda \leq C(d,N)$, uniformly in $\lambda > 1$. Fix $\lambda > 1$ and $b = (b_1,\ldots,b_m) \in \mathbb{C}^m$ such that $|b_i| \leq 1$, $1 \leq i \leq m$, and put

$$\Gamma(z,h) = \Gamma_b(z,h) = \sum_{j=1}^{d}\Gamma_b(z+h_j) = \sum_{j=1}^{d}\frac{1}{m}\sum_{i=1}^{m}\gamma(z+b_i+h_j) \tag{7.4}$$

with $h = (h_1,h_2,\ldots,h_d)$, $h_1 = 0$ and $z + b_i + h_j \in \Delta$.

Now set

$$\begin{aligned}M_\lambda(f_1,f_2,\ldots,f_d)(x) &= \prod_{j=1}^{d}(T_\lambda^{\Gamma_b} f_j)(x)\\ &= \psi(x)^d\int\int_{\Delta_{b,h}} e^{i\lambda x\cdot\Gamma(z,h)}\prod_{j=1}^{d}[f_j(z+h_j)w_b(z+h_j)]\, d\mu(z)\, d\mu(h_2)\cdots d\mu(h_d)\end{aligned}$$

where $\Delta_{b,h} = \bigcap_{j=1}^{d}\bigcap_{i=1}^{m}(\Delta - b_i - h_j)$.

As before, define the decomposed operators by

$$M_{\lambda,k}(f_1, f_2, \ldots, f_d)(x) = \psi(x)^d \int_{S_k} \int_{\Delta_{b,h}} e^{i\lambda x\cdot\Gamma(z,h)} \prod_{j=1}^{d} [f_j(z+h_j)w_b(z+h_j)]\, d\mu(z)\, d\mu(h_2)\cdots d\mu(h_d)$$

where $S_k = \{(0, h') = (0, h_2, \ldots, h_d) \in B(1)^d : 2^{-k-1} < \mathbf{v}(0, h') \le 2^{-k}\}$, $k \in \mathbb{Z}$.

Note that $\Gamma(z, h)$ may be written in the form $d \cdot (dm)^{-1} \sum_{i=1}^{dm} \gamma(z + c_i)$ for some c_i. (In fact, we may take $c_i = b_j + h_k$ with $c_1 = b_1 + h_1 = 0$ and the rest numbered in some way.) Thus, $\Gamma(z, h)$ is an offspring curve except for the factor d. To remove the d, we make the substitution $y = d \cdot x$, which dilates the support of the cutoff function by a factor d. Since $\psi(y/d)$ is bounded by the sum of $O(1)$ translates of $\psi(y)$, we may apply the definition of A_λ. This only increases the constant by a bounded factor C_d. (Moreover, observe that the new domain of integration $\Delta_{b,h}$ is in the required form: $\Delta_{b,h} = \bigcap_{i=1}^{dm}(\Delta - c_i)$ with $c_1 = 0$.)

Recall that $J_{\mathbb{C}}(z_1, \ldots, z_d)$ denotes the determinant of the holomorphic Jacobian matrix for the mapping $(z_1, \ldots, z_d) \mapsto \mathbf{\Gamma}(z_1, \ldots, z_d) = \sum_{k=1}^{d} \Gamma_b(z_k)$. Thus, Lemma 3.3 implies that

$$\begin{aligned} J_{\mathbb{R}}(z_1, \ldots, z_d) &= |J_{\mathbb{C}}(z_1, \ldots, z_d)|^2 \\ &\ge c_{d,N}\, \mathbf{v}(h)^2 \cdot \frac{1}{d} \sum_{j=1}^{d} w_b(z+h_j)^{\frac{d^2+d}{2}} \ge c_{d,N}\, \mathbf{v}(h)^2 \prod_{j=1}^{d} w_b(z+h_j)^{\frac{d+1}{2}} \end{aligned} \tag{7.5}$$

for $z \in \Delta_{b,h} = \bigcap_{j=1}^{d} \bigcap_{i=1}^{m} (\Delta - b_i - h_j)$.

We also have

$$|\tau(z,h)| = \left|\det\bigl(\Gamma'(z,h), \ldots, \Gamma^{(d)}(z,h)\bigr)\right| = \frac{1}{m} \left| \sum_{i=1}^{d} \sum_{j=1}^{m} \phi^{(d)}(z + b_j + h_i) \right|$$

where $\Gamma(z, h)$ is as in (7.4).

Thus, as in the proof of Lemma 3.3 we obtain

$$|\tau(z,h)| \ge c_{d,N} \sum_{i=1}^{d} \frac{1}{m} \sum_{j=1}^{m} |\phi^{(d)}(z + b_j + h_i)| \ge c \max_{i=1,\ldots,d} w_b(z+h_i)^{\frac{d^2+d}{4}} \tag{7.6}$$

for $z \in \Delta_{b,h}$.

The estimates (7.6) and (7.5) correspond to (6.18) (or (6.2)) and (6.13) (or (6.3)), respectively, in the proof given in Section 6. (Note that here we need to keep track of the b_i's unlike in the previous section. This is because only a weak form of a Jacobian bound, i.e. Lemma 3.3, is available in this context.)

The rest of the argument is the same as that in Section 6. □

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