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Some geometric properties of disk algebras

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ABSTRACT

In this paper we study some geometrical properties of certain classes of uniform algebras, in particular the ball algebra $\mathcal{A}_u(B_X)$ of all uniformly continuous functions on the closed unit ball and holomorphic on the open unit ball of a complex Banach space X. We prove that $\mathcal{A}_u(B_X)$ has k-numerical index 1 for every k, the lushness and also the AHSP. Moreover, the disk algebra $\mathcal{A}(\mathbb{D})$, and more in general any uniform algebra whose Choquet boundary has no isolated points, is proved to have the polynomial Daugavet property. Most of those properties are extended to the vector valued version A^X of a uniform algebra A.

1. Introduction and preliminaries

Several geometrical properties of the space of real or complex continuous functions $C(K)$ (respectively, $C(K, X)$) on a Hausdorff compact set K (respectively, with values in a Banach space X) have been obtained, related to the numerical index [11,14,16,18–20], Daugavet property [7,15,21,22], Bishop–Phelps–Bollobás property [1–3,8], etc. Most of the results at one point or another use the classical Urysohn lemma, that states that a Hausdorff topological space is normal if and only if given two disjoint closed subsets can be separated by a continuous function with values on $[0, 1]$ and taking the value 1 on one closed set and 0 on the other set.

Many of those results could not be extended to a uniform algebra (i.e. to a closed subalgebra of a complex $C(K)$ that separates points) up to now since the Urysohn lemma cannot be true, in general, if we ask the function to be in a given uniform algebra, for example its most representative case, the disk algebra $\mathcal{A}(\mathbb{D})$ of functions continuous on the closed unit complex disk $\overline{\mathbb{D}}$ and holomorphic in the open disk $\mathbb{D}$ of $\mathbb{C}$. But very recently B. Cascales, A.J. Guirao and V. Kadets in [5, Lemma 2.8 and 2.12] have constructed Urysohn type lemmas in order to extend some results on the Bishop–Phelps–Bollobás property to uniform algebras, and the functions obtained in those lemmas satisfy completely the "spirit" of "separating" the closed sets.

In this paper we are going to show that most of the aforementioned properties can be proven for uniform algebras, thanks to those Urysohn type lemmas. In particular, it is deduced from our results that the disk algebra $\mathcal{A}(\mathbb{D})$ has k-numerical index 1 for every k (Corollary 2.3), it is lush (Corollary 2.13) and has the AHSP (Corollary 2.17). Actually those results are obtained for any uniform algebra A. Moreover, $\mathcal{A}(\mathbb{D})$, and more in general any uniform algebra whose Choquet boundary has no isolated points, has the polynomial Daugavet property (Corollaries 2.9 and 2.10). Most of those properties are extended to the vector valued version A^X of a uniform algebra A.

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Here, $C(K)$ stands for the space of complex-valued continuous functions defined on a compact Hausdorff space K equipped with the supremum norm $\|\cdot\|_\infty$ and a uniform algebra is a closed subalgebra $A \subset C(K)$ that separates the points of K. We say that the uniform algebra $A \subset C(K)$ is unital if the constant function $\mathbf{1}$ belongs to A. More precisely, the complex version of Urysohn lemma is stated as follows.

Given $x \in K$, we denote by $\delta_x : A \to \mathbb{C}$ the evaluation functional at x given by $\delta_x(f) = f(x)$, for $f \in A$. The natural injection $i : K \to A^*$ defined by $i(t) = \delta_t$ for $t \in K$ is a homeomorphism from K onto $(i(K), w^*)$, where w^* denotes the weak-star topology. A set $S \subset K$ is said to be a boundary for the uniform algebra A if for every $f \in A$ there exists $x \in S$ such that $|f(x)| = \|f\|_\infty$.

For a unital uniform algebra A of $C(K)$, if

$$S = \{x^* \in A^* : \|x^*\| = 1, x^*(\mathbf{1}) = 1\},$$

then the set $\Gamma_0(A)$ of all $t \in K$ such that δ_t is an extreme point of S is a boundary for A that is called the Choquet boundary of A.

Lemma 1.1 ([5, Lemma 2.2]). *Let $A \subset C(K)$ be a unital uniform algebra for some compact Hausdorff space K and $\Gamma_0 = \Gamma_0(A)$. Then, for every open set $U \subset K$ with $U \cap \Gamma_0 \neq \phi$ and $0 < \epsilon < 1$, there exist $f \in A$ and $t_0 \in U \cap \Gamma_0$ such that $f(t_0) = \|f\|_\infty = 1$, $|f(t)| < \epsilon$ for every $t \in K \setminus U$ and $f(K) \subset R_\epsilon = \{z \in \mathbb{C} : |\mathrm{Re}(z) - 1/2| + (1/\sqrt{\epsilon})|\mathrm{Im}(z)| \leq 1/2\}$. In particular,*

$$|f(t)| + (1-\epsilon)|1 - f(t)| \leq 1, \quad \text{for all } t \in K.$$

Given an algebra A, by $m(A)$ we denote its spectrum. It is well known that $A \subset C(K)$ is a uniform algebra if and only if the Gelfand transform $\widehat{\ }: A \longrightarrow \widehat{A} \subset C(m(A), w(A^*, A))$ defined by $\widehat{a}(\phi) = \phi(a)$ for every $\phi \in m(A)$ and $a \in A$ is an injective isometry. Moreover, if A is a unital uniform algebra, then $\widehat{A}$ is also a unital uniform algebra.

For a non-unital uniform algebra $B \subset C(S)$, that is, a uniform algebra without $\mathbf{1}$, unless otherwise stated in the remainder of this paper, we let $K = (m(B), w^*)$ and identify B with a subalgebra of $C(m(B))$ by using the Gelfand transform. We denote by $A = \{c\mathbf{1} + f : c \in \mathbb{C}, f \in B\}$ the $\|\cdot\|_\infty$-closed subalgebra generated by $B \cup \{\mathbf{1}\}$. Consider the Choquet boundary $\Gamma_0(A)$. Since B is a maximal ideal of A, the Gelfand–Mazur theorem ensures that there exists $v \in K$ such that $B = \{f \in A : \delta_v = 0\}$. Denote $\Gamma_0 = \Gamma_0(A) \setminus \{v\}$, then Γ_0 is a boundary for B.

Lemma 1.2 ([5, Lemma 2.12]). *Let B be a non-unital uniform algebra and consider $K = m(B)$. Then, for every open set $U \subset K$ with $U \cap \Gamma_0 \neq \phi$ and $0 < \epsilon < 1$, there exist $f \in B$ and $t_0 \in U \cap \Gamma_0$ such that $f(t_0) = \|f\|_\infty = 1$, $|f(t)| < \epsilon$ for every $t \in K \setminus U$ and*

$$|f(t)| + (1-\epsilon)|1 - f(t)| \leq 1, \quad \text{for all } t \in K.$$

We apply this Urysohn type lemma to extend results on $C(K)$ or $C(K, X)$ to A^X concerning the numerical index, Daugavet equation, lushness and the approximate hyperplane series property (in short *AHSP*). If A is a uniform algebra, then A^X is defined to be a subspace of $C(K, X)$ such that

$$A^X = \{f \in C(K, X) : x^* \circ f \in A \text{ for all } x^* \in X^*\}.$$

Given $f \in A$ and $x \in X$, we define $f \otimes x \in C(K, X)$ by $(f \otimes x)(t) = f(t)x$ for $t \in K$. We write

$$A \otimes X = \{f \otimes x;\ f \in A,\ x \in X\}.$$

From the definition of A^X we note that $A \otimes X \subset A^X$.

For a Banach space X, we write $\Pi(X)$ to denote the subset of $X \times X^*$ given by

$$\Pi(X) := \left\{(x, x^*) : x \in S_X,\ x^* \in S_{X^*},\ x^*(x) = 1\right\}.$$

Given a bounded function $\Phi : S_X \to X$, its *numerical range* is defined by

$$V(\Phi) := \{x^*(\Phi(x)) : (x, x^*) \in \Pi(X)\}$$

and its *numerical radius* is defined by

$$v(\Phi) := \sup\{|\lambda| : \lambda \in V(\Phi)\}.$$

Let us comment that for a bounded function $\Phi : \Omega \to X$, where $S_X \subset \Omega \subset X$, the above definitions are applied by just considering $V(\Phi) := V(\Phi|_{S_X})$.

For $k \in \mathbb{N}$, we define

$$n^{(k)}(X) = \inf\{v(P) : P \in \mathcal{P}(^kX; X),\ \|P\| = 1\},$$

where $\mathcal{P}(^kX; X)$ is the space of all continuous k-homogeneous polynomials from X into X, and call it the *polynomial numerical index of order k of X* [6]. When $k = 1$, it is the numerical index of X which was first suggested by G. Lumer in 1968.

Given two complex Banach spaces X and Y, by $\mathcal{A}_u(B_X, Y)$ we denote the space of all uniformly continuous mappings on the closed unit ball $\overline{B}_X$ and holomorphic on the open unit ball B_X of X with values in Y. If X is finite dimensional since the continuous mappings on the closed unit ball $\overline{B}_X$ are uniformly continuous then we will use the standard notation $\mathcal{A}(B_X, Y)$ instead of $\mathcal{A}_u(B_X, Y)$.

If we consider elements of $\mathcal{A}_u(B_X, X)$ instead of continuous k-homogeneous polynomials, we can define, as done in [17], the *analytic numerical index of X* by

$$n_a(X) = \inf\{v(f) : f \in \mathcal{A}_u(B_X, X),\ \|f\| = 1\}.$$

Since the space $\mathcal{P}(X; X)$ of all continuous polynomials from X into X is dense in $\mathcal{A}_u(B_X, X)$ we have that

$$n_a(X) = \inf\{v(P) : P \in \mathcal{P}(X; X),\ \|P\| = 1\},$$

i.e. $n_a(X)$ can be called the "*non-homogeneous polynomial numerical index of X*". Clearly,

$$n_a(X) \le n^{(k)}(X)$$

for every $k \in \mathbb{N}$. We also denote by $\mathcal{P}(X)$ the space of scalar-valued polynomials on X.

2. The results

We first study the polynomial numerical index of order k of A^X.

Theorem 2.1. *Suppose that A is a uniform algebra. Then $n^{(k)}(A^X) \ge n^{(k)}(X)$ for every $k \ge 1$ and $n_a(A^X) \ge n_a(X)$.*

Proof. We prove only $n^{(k)}(X) \le n^{(k)}(A^X)$ for every $k \ge 1$, because the other case can be proved in the same way. We assume that $n^{(k)}(X) > 0$, since otherwise the result is trivial.

Let $P \in S_{\mathcal{P}(^kA^X, A^X)}$ and $0 < \epsilon < 1$ be given. Choose $f_0 \in S_{A^X}$ so that $\|P(f_0)\| > 1 - \frac{\epsilon}{6}$. Find $t_1 \in \Gamma_0$ such that $\big\|P(f_0)(t_1)\big\| > 1 - \frac{\epsilon}{6}$. Since P is continuous at f_0, there exists $0 < \delta < 1$ such that $\|P(f_0) - P(g)\| < \frac{\epsilon}{6}$ for every $g \in A^X$ with $\|f_0 - g\| < \delta$.

Let

$$W = \{t \in K : \|f_0(t) - f_0(t_1)\| < \delta/6,\ \|P(f_0)(t) - P(f_0)(t_1)\| < \epsilon/3\}.$$

This set is open in K and $t_1 \in W \cap \Gamma_0$.

From Lemmas 1.1 and 1.2, there exist a function $\phi : K \to \overline{\mathbb{D}}$ and $t_0 \in W \cap \Gamma_0$ such that $\phi \in A$, $\phi(t_0) = 1$, $|\phi(w)| < \frac{\delta}{6}$ for every $w \in K \setminus W$, and

$$|\phi(t)| + \left(1 - \frac{\delta}{6}\right)|1 - \phi(t)| \le 1$$

for every $t \in K$.

Define $\Psi : X \to A^X$ by $\Psi(x) = \big(1 - \frac{\delta}{6}\big)(\mathbf{1} - \phi)f_0 + \phi x$ for all $x \in X$.

We note that if $\mu \in A$ and $g \in A^X$, then $\mu g \in A^X$. In fact, $[x^* \circ (\mu g)](t) = \mu(t)x^*(g(t)) = \big(\mu(x^* \circ g)\big)(t)$ and $\mu(x^* \circ g) \in A$ for $x^* \in X^*$.

Therefore, Ψ is well-defined and $\|\Psi(x)\| \le 1$ for every $x \in B_X$. Put $x_0 = f_0(t_0) \in B_X$.

Then,

$$\begin{aligned}\big\|f_0 - \Psi(x_0)\big\| &= \sup_{t\in K}\left\|f_0(t) - \left(1 - \frac{\delta}{6}\right)\big(1 - \phi(t)\big)f_0(t) - \phi(t)f_0(t_0)\right\| \\ &\le \sup_{t\in K}\left(\frac{\delta}{6}\|f_0(t)\| + |\phi(t)|\|f_0(t) - f_0(t_0)\| + \frac{\delta}{6}|\phi(t)|\|f_0(t)\|\right) \\ &< \frac{\delta}{6} + \frac{\delta}{3} + \frac{\delta}{6} = \delta.\end{aligned}$$

Hence,

$$\begin{aligned}\big\|P(\Psi(x_0))(t_0)\big\| &\ge \big\|P(\Psi(x_0))(t_1)\big\| - \big\|P(\Psi(x_0))(t_1) - P(\Psi(x_0))(t_0)\big\| \\ &\ge \big\|P(f_0)(t_1)\big\| - \big\|P(f_0)(t_1) - P(\Psi(x_0))(t_1)\big\| - \big\|P(f_0)(t_1) - P(f_0)(t_0)\big\| \\ &\quad - \big\|P(\Psi(x_0))(t_1) - P(f_0)(t_1)\big\| - \big\|P(f_0)(t_0) - P(\Psi(x_0))(t_0)\big\| \\ &> 1 - \frac{\epsilon}{6} - \frac{\epsilon}{6} - \frac{\epsilon}{3} - \frac{\epsilon}{6} - \frac{\epsilon}{6} = 1 - \epsilon.\end{aligned}$$

Choose $x_0^* \in S_{X^*}$ so that

$$x_0^*\big[P\big(\Psi(x_0)\big)(t_0)\big] > 1-\epsilon.$$

Write $x_0 = z_0\tilde{x}_0$ for suitable $z_0 \in \overline{\mathbb{D}}$ and $\tilde{x}_0 \in S_X$.

Consider an entire function

$$z \in \mathbb{C} \longrightarrow x_0^*\big[P\big(\Psi(z\tilde{x}_0)\big)(t_0)\big] \in \mathbb{C}.$$

By the maximum modulus theorem, there exists z_1, $|z_1| = 1$ where the above function attains its maximum on $\overline{\mathbb{D}}$. Hence,

$$\begin{aligned}\big\|P\big(\Psi(z_1\tilde{x}_0)\big)(t_0)\big\| &\ge \big|x_0^*\big[P\big(\Psi(z_1\tilde{x}_0)\big)(t_0)\big]\big| \\ &\ge \big|x_0^*\big[P\big(\Psi(z_0\tilde{x}_0)\big)(t_0)\big]\big| > 1-\epsilon.\end{aligned}$$

Put $x_1 = z_1\tilde{x}_0 \in S_X$ and consider $x_1^* \in S_{X^*}$ with $x_1^*(x_1) = 1$.

Define $\Phi(x) = x_1^*(x)\big(1-\frac{\delta}{6}\big)(1-\phi)f_0 + \phi x \in A^X$ for $x \in X$.

Note that $\Phi(x_1) = \Psi(x_1) = \Psi(z_1\tilde{x}_0)$, hence

$$\big\|P\big(\Phi(x_1)\big)(t_0)\big\| > 1-\epsilon.$$

Define $Q \in \mathcal{P}({}^kX; X)$ by

$$Q(x) = P\big(\Phi(x)\big)(t_0), \quad (x \in X). \tag{2.1}$$

Then $1 \ge \|Q\| \ge \|Q(x_1)\| = \big\|P\big(\Phi(x_1)\big)(t_0)\big\| > 1-\epsilon$.

For $0 < \epsilon < n^{(k)}(X)$, we can choose $(x_2, x_2^*) \in \Pi(X)$ so that

$$|x_2^*(Q(x_2)/\|Q\|)| > v(Q/\|Q\|) - \epsilon \ge n^{(k)}(X) - \epsilon > 0.$$

Then $|x_2^*(Q(x_2))| > \|Q\|(n^{(k)}(X)-\epsilon) > (1-\epsilon)(n^{(k)}(X)-\epsilon)$.

Note that

$$(\Phi(x_2), x_2^* \circ \delta_{t_0}) \in \Pi(A^X) \quad \text{because } \Phi(x_2)(t_0) = x_2.$$

$$\begin{aligned} v(P) &\ge \big|(x_2^* \circ \delta_{t_0})(P(\Phi(x_2)))\big| = \big|x_2^*\big([P(\Phi(x_2))](t_0)\big)\big| \\ &= |x_2^*(Q(x_2))| > (1-\epsilon)(n^{(k)}(X)-\epsilon).\end{aligned}$$

Hence, we get that $n^{(k)}(A^X) \ge (1-\epsilon)\big(n^{(k)}(X)-\epsilon\big)$. Since ϵ is arbitrary, $n^{(k)}(A^X) \ge n^{(k)}(X)$. In the above, we do not use the homogeneity of k-homogeneous polynomials. Therefore, it is true for the analytic numerical index. □

Theorem 2.2. *Let A be a uniform algebra and X be a Banach space. Assume that A^X has the following property: For every $P \in \mathcal{P}({}^kX; X)$ and $t \in K$, $Q : A^X \longrightarrow C(K, X)$ where $Q(f)(t) = P(f(t))$ satisfies that $Q(f) \in A^X$ for every $f \in A^X$. Then $n^{(k)}(A^X) = n^{(k)}(X)$.*

Proof. By Theorem 2.1 we only have to prove that $n^{(k)}(A^X) \le n^{(k)}(X)$.

Consider

$$L = \big\{(f, x^* \circ \delta_t) : f \in S_{A^X},\ t \in K,\ x^* \in S_{X^*} \text{ and } x^*(f(t)) = 1\big\}.$$

For the projection $\pi_1 : A^X \times (A^X)^* \to A^X$ we have $\pi_1(L) = S_{A^X}$, hence for every $Q \in \mathcal{P}({}^kA^X; A^X)$ we can see that

$$v(Q) = \sup\big\{\big|x^*(Q(f)(t))\big| : (f, x^* \circ \delta_t) \in L\big\} \quad \text{(see [12])}.$$

Let $P \in \mathcal{P}({}^kX; X)$, $\|P\| = 1$. Define $Q \in \mathcal{P}({}^kA^X; A^X)$ by $Q(f)(t) = P(f(t))$. Then $\|Q\| = 1$ and $v(Q) \ge n^{(k)}(A^X)$. For every $\epsilon > 0$, there exists $(f, x^* \circ \delta_t) \in L$ such that

$$n^{(k)}(A^X) - \epsilon \le v(Q) - \epsilon < \big|(x^* \circ \delta_t)Q(f)\big| = \big|x^*(P(f(t)))\big| \le v(P),$$

therefore $n^{(k)}(A^X) \le n^{(k)}(X)$. □

We do not use the homogeneity of k-homogeneous polynomials in the above proof. Hence, $n_a(A^X) \le n_a(X)$ under the similar condition for every $F \in \mathcal{A}_u(B_X, X)$ and $t \in K$ to that of Theorem 2.2.

Corollary 2.3. *For any Banach space X, and $k \ge 1$, the following hold.*

(1) *For a uniform algebra A we have $n^{(k)}(A) = 1$ for every $k \ge 1$ and $n_a(A) = 1$.*
(2) *$n^{(k)}(\mathcal{A}(\mathbb{D}^n, X)) = n^{(k)}(X)$ and $n_a(\mathcal{A}(\mathbb{D}^n, X)) = n_a(X)$ for every $n \in \mathbb{N}$.*
(3) *$n^{(k)}\big(\mathcal{A}_u(B_X)\big) = 1$ for every $k \ge 1$ and $n_a\big(\mathcal{A}_u(B_X)\big) = 1$.*

Proof. (1) follows from Theorem 2.1 and the fact that $n^{(k)}(\mathbb{C}) = 1$ for every $k \geq 1$. (2) follows from Theorems 2.1 and 2.2, because $\mathcal{A}(\mathbb{D}^n, X) = \mathcal{A}(\mathbb{D}^n)^X$. We note here that every weakly holomorphic function is holomorphic. (3) follows from the fact that $\mathcal{A}_u(B_X)$ is identified with a uniform algebra $\widehat{\mathcal{A}_u(B_X)}$ of $C(K)$, where

$$K = \left(\overline{\{\delta_x : x \in B_X\}}^{w^*}, w\left(\mathcal{A}_u(B_X)^*, \mathcal{A}_u(B_X)\right)\right)$$

by using the Gelfand transform. □

By $\mathcal{A}_{w^*u}(B_{Y^*}, X)$ we denote the space of all w^*-uniformly continuous mappings on the closed unit ball and holomorphic on the open unit ball of the dual Y^* of a complex Banach space Y with values in X and by $\mathcal{A}_{wu}(B_Y, X)$ we denote the space of all weakly uniformly continuous mappings on the closed unit ball and holomorphic on the open unit ball of a complex Banach space Y with values in X.

Corollary 2.4. *Let X and Y be Banach spaces.*

(1) $n^{(k)}\left(\mathcal{A}_{w^*u}(B_{Y^*}, X)\right) = n^{(k)}(X)$ *for every* $k \geq 1$ *and* $n_a\left(\mathcal{A}_{w^*u}(B_{Y^*}, X)\right) = n_a(X)$.
(2) $n^{(k)}\left(\mathcal{A}_{wu}(B_Y, X)\right) = n^{(k)}(X)$ *for every* $k \geq 1$ *and* $n_a\left(\mathcal{A}_{wu}(B_Y, X)\right) = n_a(X)$.

Proof. (1) We note that $(\overline{B}_{Y^*}, w^*)$ is compact and $\mathcal{A}_{w^*u}(B_{Y^*}, X) = \mathcal{A}_{w^*u}(B_{Y^*})^X$.

(2) It is enough to see that $\mathcal{A}_{wu}(B_Y, X)$ is isometrically isomorphic to $\mathcal{A}_{w^*u}(B_{Y^{**}}, X)$. For each $f \in C_{wu}(\overline{B}_Y, X)$ we find a unique extension $\widetilde{f} \in C_{w^*u}(\overline{B}_{Y^{**}}, X)$ of f, because $\overline{B}_Y$ is w^* dense in $\overline{B}_{Y^{**}}$ and X is complete. Since it is obvious that if g is a polynomial, then $\widetilde{g}$ is also a polynomial, and the weakly continuous polynomials on the open unit ball are dense in $\mathcal{A}_{wu}(B_Y, X)$ we get that $\widetilde{f} \in \mathcal{A}_{w^*u}(B_{Y^{**}}, X)$ for every $f \in \mathcal{A}_{wu}(B_Y, X)$. Conversely, for any $g \in \mathcal{A}_{w^*u}(B_{Y^{**}}, X)$ we can see that $g|_{B_Y} \in \mathcal{A}_{wu}(B_Y, X)$, and so $\widetilde{g|_{B_Y}} = g$ by the uniqueness. Hence the mapping $T : \mathcal{A}_{wu}(B_Y, X) \longrightarrow \mathcal{A}_{w^*u}(B_{Y^{**}}, X)$ defined by $T(f) = \widetilde{f}$ is an isometric isomorphism. □

Remark 2.5. Corollary 2.4 actually shows that if X or Y is finite dimensional, then $n^{(k)}\left(\mathcal{A}_u(B_Y, X)\right) = n^{(k)}(X)$ for every $k \geq 1$ and $n_a\left(\mathcal{A}_u(B_Y, X)\right) = n_a(X)$. If Y is finite dimensional, we note that $\mathcal{A}_u(B_Y, X) = \mathcal{A}_{wu}(B_Y, X)$.

Suppose that X is n-dimensional. If we identify X with $\mathbb{C}^n$ then $g = (g_1, \ldots, g_n) \in \widehat{\mathcal{A}_u(B_Y)}^X$ if and only if $g \in C(m(\mathcal{A}_u(B_Y)), X)$ and $x^* \circ g \in \widehat{\mathcal{A}_u(B_Y)}$ for every $x^* \in X^*$, and this happens if and only if $g_1, \ldots, g_n \in \widehat{\mathcal{A}_u(B_Y)}$. Hence $g_j = \widehat{f_j}, f_j \in \mathcal{A}_u(B_Y)$ for every $j = 1, \ldots, n$. Clearly the mapping $T : \widehat{\mathcal{A}_u(B_Y)}^X \longrightarrow \mathcal{A}_u(B_Y, X)$ defined by $T(g) = (f_1, \ldots, f_n)$ provides an isometric isomorphism.

The problem appears when X and Y are both infinite dimensional. In this case it is not clear at all that $\mathcal{A}_u(B_Y, X)$ can be included in $C(K, X)$ for a suitable compact set K.

We next see the application of the Urysohn type lemmas to show the polynomial Daugavet property of A^X. A Banach space X is said to have the Daugavet property if the norm identity, the so called Daugavet equation,

$$\|Id + T\| = 1 + \|T\|$$

holds for every rank-one operator (and hence for every weakly compact operator) $T \in L(X)$. If this happens for every weakly compact polynomial, we say that X has the polynomial Daugavet property.

Lemma 2.6 ([7, Proposition 1.3 and Corollary 2.2]). *Let X be a real or complex Banach space. Then the following are equivalent.*

(1) *X has the polynomial Daugavet property, that is, every weakly compact $P \in \mathcal{P}(X; X)$ satisfies the Daugavet equation.*
(2) *For every $p \in \mathcal{P}(X)$ with $\|p\| = 1$, every $x_0 \in S_X$, and every $\epsilon > 0$, there exist $\omega \in \mathbb{T}$ and $y \in B_X$ such that*

$$\mathrm{Re}\,\omega p(y) > 1 - \epsilon \quad \text{and} \quad \|x_0 + \omega y\| > 2 - \epsilon.$$

(3) *Every weakly compact $P \in \mathcal{P}(X, X)$ satisfies* $\sup \mathrm{Re}\,V(P) = \|P\|$.

The next theorem shows that a result by Werner [21, Theorem 3.3] on the Daugavet equation is actually true for the polynomial Daugavet equation.

Theorem 2.7. *Suppose that A is a uniform algebra whose Choquet boundary has no isolated points. For every $P \in S_{\mathcal{P}(A^X)}$, $f_0 \in S_{A^X}$ and $\epsilon > 0$, there exist some $\omega \in S_{\mathbb{C}}$ and $g \in B_{A^X}$ such that* $\mathrm{Re}\,\omega P(g) > 1 - \epsilon$ *and* $\|f_0 + \omega g\| > 2 - \epsilon$.

Proof. Let $0 < \epsilon < 1$ be given, and fix $P \in S_{\mathcal{P}(A^X)}$ and $f_0 \in S_{A^X}$.

Choose $h \in S_{A^X}$ so that $|P(h)| > 1 - \epsilon/2$, and $|\omega| = 1$ such that $\mathrm{Re}\,\omega P(h) = |P(h)| > 1 - \epsilon/2$. Choose $t_0 \in \Gamma_0$ so that $\|f_0(t_0)\| > 1 - \epsilon/8$.

Let

$$U = \{t \in K : \|f_0(t) - f_0(t_0)\| < \epsilon/8,\ \|h(t) - h(t_0)\| < \epsilon/8\}.$$

We have two possible cases.

Case 1.

There exists a sequence $(t_i)_{i=1}^{\infty} \subset U$ such that $\|\omega^{-1}f_0(t_i) - h(t_i)\|$ goes to 0. Then, we get

$$\begin{aligned}\|f_0 + \omega h\| &\geq \sup_i \|f_0(t_i) + \omega h(t_i)\| \\ &\geq \sup_i (2\|f_0(t_0)\| - 2\|f_0(t_0) - f_0(t_i)\| - \|f_0(t_i) - \omega h(t_i)\|) \\ &\geq 2 - \frac{\epsilon}{4} - \frac{\epsilon}{4} - \frac{\epsilon}{4} > 2 - \epsilon\end{aligned}$$

which completes our proof.

Case 2.

There exists $\alpha > 0$ such that $\|\omega^{-1}f_0(t) - h(t)\| > \alpha$ for every $t \in U$. Since t_0 is not an isolated point of Γ_0 and belongs to the open set U, there exist nonempty disjoint open subsets $U_i \subset K$ such that $\cup_{i=1}^{\infty} U_i \subset U$ and $U_i \cap \Gamma_0 \neq \emptyset$ for every i. Thus Lemmas 1.1 and 1.2 imply that there exist $\phi_i \in A$ and $t_i \in \Gamma_0 \cap U_i$ such that $\phi_i(t_i) = \|\phi_i\|_\infty = 1$, $|\phi_i(t)| < \frac{\epsilon}{2^{i+3}}$ for every $t \in U_i^c$ and $\big(1 - \frac{\epsilon}{2^{i+3}}\big)|1 - \phi_i(t)| + |\phi_i(t)| \leq 1$ for every $t \in K$.

Put

$$\tilde{h}_i = h + \phi_i\big(\omega^{-1}f_0(t_i) - h(t_i)\big) \in A^X.$$

For every $t \in \cup_{i=1}^{\infty} U_i$, we get that

$$\begin{aligned}\|\tilde{h}_i(t)\| &= \|h(t) + \phi_i(t)\omega^{-1}f_0(t_i) - \phi_i(t)h(t_i)\| \\ &\leq \|h(t) - h(t_i)\| + \|h(t_i) - \phi_i(t)h(t_i)\| + \|\phi_i(t)\omega^{-1}f_0(t_i)\| \\ &\leq \|h(t) - h(t_0)\| + \|h(t_0) - h(t_i)\| + |1 - \phi_i(t)| + |\phi_i(t)| \\ &< \frac{\epsilon}{4} + \left(1 - \frac{\epsilon}{2^{i+3}}\right)|1 - \phi_i(t)| + |\phi(t)| + \frac{\epsilon}{2^{i+3}}|1 - \phi_i(t)| \\ &\leq \frac{\epsilon}{4} + 1 + \frac{\epsilon}{2^{i+2}} < 1 + \frac{\epsilon}{2}.\end{aligned}$$

For every $t \in K \setminus \cup_{i=1}^{\infty} U_i$

$$\begin{aligned}\|\tilde{h}_i(t)\| &\leq \|h(t)\| + |\phi_i(t)|\big\|\omega^{-1}f_0(t_i) - h(t_i)\big\| \\ &\leq 1 + \frac{\epsilon}{2^{i+2}} < 1 + \frac{\epsilon}{2}.\end{aligned}$$

Moreover,

$$\begin{aligned}\|\tilde{h}_i\| &\geq \|\tilde{h}_i(t_i)\| = \|\omega^{-1}f_0(t_i)\| = \|f_0(t_i)\| \\ &\geq \|f_0(t_0)\| - \|f_0(t_0) - f_0(t_i)\| \\ &\geq 1 - \frac{\epsilon}{8} - \frac{\epsilon}{8} > 1 - \frac{\epsilon}{2}.\end{aligned}$$

Put $g_i = \tilde{h}_i/\|\tilde{h}_i\|$. We get

$$\|\tilde{h}_i - g_i\| = \big|1 - \|\tilde{h}_i\|\big| \leq \frac{\epsilon}{2}.$$

On the other hand, we see that for any $(\beta_i) \in \ell_\infty$

$$\begin{aligned}\sup_n \left\|\sum_{i=1}^{n} \beta_i\phi_i\big(\omega^{-1}f_0(t_i) - h(t_i)\big)\right\| &\leq \sup_n \sup_{t\in K} \sum_{i=1}^{n} 2|\beta_i|\big\|\phi_i(t)\big\| \\ &\leq 2\left(1 + \frac{\epsilon}{8}\right)\sup_i |\beta_i|.\end{aligned}$$

It follows from [9, Theorem V.6] that $\sum_{i=1}^{\infty} \phi_i\big(\omega^{-1}f_0(t_i) - h(t_i)\big)$ is weakly unconditionally Cauchy.

Since $\big\|\phi_k(\omega^{-1}f_0(t_k) - h(t_k))\big\| > \alpha$ for every $k \in \mathbb{N}$, the Bessaga–Pełczyński selection principle allows us to extract a basic subsequence $\big(\phi_{\sigma(k)}(\omega^{-1}f_0(t_{\sigma(k)}) - h(t_{\sigma(k)}))\big)$, which is equivalent to the unit vector basis of c_0 [9, p. 45]. It then follows from the weak continuity of polynomials on a bounded subset of real or complex c_0 that $\mathrm{Re}\,\omega P(\tilde{h}_{\sigma(k)})$ converges to $\mathrm{Re}\,\omega P(h)$ [10, Proposition 1.59] as $k \to \infty$.

Choose $i \in \mathbb{N}$ so that $\mathrm{Re}\,\omega P(\tilde{h}_i) > 1 - \epsilon/2$. It follows that $\mathrm{Re}\,\omega P(g_i) > 1 - \epsilon$.
Finally,

$$\begin{aligned}\|f_0 + \omega g_i\| &\geq \|f_0 + \omega \tilde{h}_i\| - \|g_i - \tilde{h}_i\| \\ &\geq \|f_0(t_i) + \omega \tilde{h}_i(t_i)\| - \epsilon/2 = 2\|f_0(t_i)\| - \epsilon/2 \\ &\geq 2\|f_0(t_0)\| - 2\|f_0(t_0) - f_0(t_i)\| - \epsilon/2 \\ &> 2 - \frac{\epsilon}{4} - \frac{\epsilon}{4} - \frac{\epsilon}{2} = 2 - \epsilon. \quad \square\end{aligned}$$

Corollary 2.8. *The argument of the above proof is true for any closed subspace B of A^X with $A \bigotimes X \subset B$, that is, every weakly compact $P \in \mathcal{P}(B; B)$ satisfies the Daugavet equation which means that B has the polynomial Daugavet property.*

Corollary 2.9. *If A is a uniform algebra whose Choquet boundary has no isolated points, then every weakly compact $P \in \mathcal{P}(A^X; A^X)$ satisfies the Daugavet equation which means that A^X has the polynomial Daugavet property.*

Since given a ball U in $\mathbb{C}^n$ or the polydisk $\mathbb{D}^n$ the Choquet boundary of $\mathcal{A}(U)$ does not have isolated points, we have the following corollary, that extends a result by Wojtaszczyk [23, Remark after Corollary 3].

Corollary 2.10. *If U is a ball in $\mathbb{C}^n$ or the polydisk $\mathbb{D}^n$, then $\mathcal{A}(U, X)$ has the polynomial Daugavet property for every complex Banach space X. In particular, $\mathcal{A}(U)$ has the polynomial Daugavet property.*

Theorem 2.11. *Let A be a uniform algebra and X be a Banach space with the polynomial Daugavet property. Then A^X has the polynomial Daugavet property.*

Proof. Let $P \in \mathcal{P}(A^X; A^X)$ be a weakly compact polynomial. We follow the proof of Theorem 2.1. Define $Q \in \mathcal{P}(X; X)$ to satisfy the Eq. (2.1) and use the same notations as in Theorem 2.1. Since $Q \in \mathcal{P}(X; X)$ is a weakly compact polynomial, we apply Lemma 2.6 and get that $\sup \mathrm{Re}V(Q) = \|Q\| > 1 - \epsilon$.
Choose $(x_2, x_2^*) \in \Pi(X)$ so that $\mathrm{Re}x_2^*(Q(x_2)) > \|Q\| - \epsilon > 2 - \epsilon$.
Note that

$$(\Phi(x_2), x_2^* \circ \delta_{t_0}) \in \Pi(A^X) \quad \text{and} \quad \|Q\| > 1 - \epsilon.$$

We get

$$\begin{aligned}\mathrm{Re}(x_2^* \circ \delta_{t_0})(P(\Phi(x_2))) &= \mathrm{Re}x_2^*([P(\Phi(x_2))](t_0)) \\ &= \mathrm{Re}x_2^*(Q(x_2)) > 1 - 2\epsilon.\end{aligned}$$

Since ϵ is arbitrary, we get $\sup \mathrm{Re}V(P) = 1 = \|P\|$. This implies that A^X has the polynomial Daugavet property. $\square$

The concept of lushness was introduced to study an infinite dimensional Banach space with the numerical index 1 [4]. The fact that a Banach space X has numerical index 1 means that the norm of any operator on X is the same as its numerical radius. Lushness has been known to be the weakest among quite a few isometric properties in the literature which are sufficient conditions for a Banach space to have the numerical index 1.

A Banach space X is said to be *lush* if for every $x, y \in S_X$ and for every $\epsilon > 0$ there is a slice

$$S = S(B_X, x^*, \epsilon) = \{x \in B_X : \mathrm{Re}x^*(x) > 1 - \epsilon\}, \quad x^* \in S_{X^*}$$

such that $x \in S$ and $\mathrm{dist}(y, \mathrm{aconv}(S)) < \epsilon$.

Theorem 2.12. *Suppose that A is a uniform algebra.*

(1) *If X is lush, then A^X is lush. In particular, A is lush.*
(2) *If A is unital and A^X is lush, then X is lush.*

Proof. (1) Assume that X is lush. By using [13, Proposition 2.1], it is enough to show that for every $f \in S_{A^X}$, $g \in B_{A^X}$, $n \in \mathbb{N}$ and $\epsilon > 0$, there exist positive numbers $(\lambda_k)_{k=1}^n$, $\sum \lambda_k = 1$ and functions $(f_k)_{k=1}^n \subset B_{A^X}$ such that

$$\left\| f + \sum_{k=1}^n f_k \right\| > n + 1 - \epsilon, \qquad \left\| g - \sum_{k=1}^n \lambda_k e^{2\pi ik/n} f_k \right\| \leq \epsilon + \frac{2\pi}{n}.$$

Choose $t_1 \in \Gamma_0$ so that $f(t_1) \in S_X$. Apply [13, Proposition 2.1] to $f(t_1)$ and $g(t_1)$ and get positive numbers $(\lambda_k)_{k=1}^n$, $\sum_{k=1}^n \lambda_k = 1$ and vectors $(x_k)_{k=1}^n \subset B_X$ such that

$$\left\| f(t_1) + \sum_{k=1}^n x_k \right\| > n + 1 - \frac{\epsilon}{3}, \qquad \left\| g(t_1) - \sum_{k=1}^n \lambda_k e^{2\pi ik/n} x_k \right\| \leq \frac{\epsilon}{3} + \frac{2\pi}{n}.$$

Set

$$U=\{t\in K: \|g(t)-g(t_1)\|<\epsilon/3,\ \|f(t)-f(t_1)\|<\epsilon/3\}.$$

Apply Lemmas 1.1 and 1.2 to find $\phi\in A$ and $t_2\in\Gamma_0\cap U$ such that $\phi(t_2)=\|\phi\|_\infty=1$, $|\phi(t)|<\epsilon/6$ for every $t\in K\setminus U$ and

$$|\phi(t)|+\left(1-\frac{\epsilon}{6}\right)|1-\phi(t)|\le 1,\quad \text{for all } t\in K.$$

For each $1\le k\le n$ define

$$f_k(t)=\phi(t)x_k+\left(1-\frac{\epsilon}{6}\right)(1-\phi(t))e^{-2\pi ik/n}g(t)\in B_{A^X}.$$

We can see that

$$\begin{aligned}\left\|f+\sum_{k=1}^n f_k\right\| &\ge \left\|f(t_2)+\sum_{k=1}^n f_k(t_2)\right\|=\left\|f(t_2)+\sum_{k=1}^n x_k\right\|\\ &\ge \left\|f(t_1)+\sum_{k=1}^n x_k\right\|-\|f(t_1)-f(t_2)\|\\ &> n+1-\epsilon.\end{aligned}$$

For $t\in U$, we get from $\|g(t)-g(t_1)\|<\epsilon/3$ that

$$\begin{aligned}\left\|g(t)-\sum_{k=1}^n\lambda_k e^{2\pi ik/n}f_k(t)\right\| &= \left\|\phi(t)\left(g(t)-\sum_{k=1}^n\lambda_k e^{2\pi ik/n}x_k\right)+\frac{\epsilon}{6}(1-\phi(t))g(t)\right\|\\ &\le \left\|\left(g(t)-\sum_{k=1}^n\lambda_k e^{2\pi ik/n}x_k\right)\right\|+\left\|\frac{\epsilon}{6}(1-\phi(t))g(t)\right\|\\ &\le \|g(t)-g(t_1)\|+\left\|g(t_1)-\sum_{k=1}^n\lambda_k e^{2\pi ik/n}x_k\right\|+\epsilon/3\\ &\le \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{2\pi}{n}+\frac{\epsilon}{3}=\epsilon+\frac{2\pi}{n},\end{aligned}$$

and for $t\in K\setminus U$ we get

$$\begin{aligned}\left\|g(t)-\sum_{k=1}^n\lambda_k e^{2\pi ik/n}f_k(t)\right\| &= \left\|\phi(t)\left(g(t)-\sum_{k=1}^n\lambda_k e^{2\pi ik/n}x_k\right)+\frac{\epsilon}{6}\big(1-\phi(t)\big)g(t)\right\|\\ &\le |\phi(t)|\left\|\left(g(t)-\sum_{k=1}^n\lambda_k e^{2\pi ik/n}x_k\right)\right\|+\left\|\frac{\epsilon}{6}(1-\phi(t))g(t)\right\|\\ &\le \frac{\epsilon}{3}+\frac{\epsilon}{3}<\epsilon.\end{aligned}$$

(2) Assume that A is unital and A^X is lush. Similarly to the proof of (i), it is enough to show that for every $x\in S_X$, $y\in B_X$, $n\in\mathbb{N}$ and $\epsilon>0$, there exist positive numbers $(\lambda_k)_{k=1}^n$, $\sum_{k=1}^n\lambda_k=1$ and vectors $(x_k)_{k=1}^n\subset B_X$ such that

$$\left\|x+\sum_{k=1}^n x_k\right\|>n+1-\epsilon,\qquad \left\|y-\sum_{k=1}^n\lambda_k e^{2\pi ik/n}x_k\right\|\le\epsilon+\frac{2\pi}{n}.$$

Since A is unital, we can define $f\in S_{A^X}$ and $g\in B_{A^X}$ by $f(t)=x$ and $g(t)=y$ for every $t\in K$.

Since A^X is lush, we apply [13, Proposition 2.1] to f and g to get positive numbers $(\lambda_k)_{k=1}^n$, $\sum\lambda_k=1$ and functions $(f_k)_{k=1}^n\subset B_{A^X}$ such that

$$\left\|f+\sum_{k=1}^n f_k\right\|>n+1-\epsilon,\qquad \left\|g-\sum_{k=1}^n\lambda_k e^{2\pi ik/n}f_k\right\|\le\epsilon+\frac{2\pi}{n}.$$

Choose t_1 so that

$$\left\|f(t_1)+\sum_{k=1}^n f_k(t_1)\right\|>n+1-\epsilon,$$

then

$$\left\|x+\sum_{k=1}^{n} f_k(t_1)\right\| > n+1-\epsilon, \qquad \left\|y-\sum_{k=1}^{n} \lambda_k e^{2\pi ik/n} f_k(t_1)\right\| \le \epsilon + \frac{2\pi}{n}. \quad \square$$

Corollary 2.13. *If X is lush, then for any Banach space Y*

(1) $\mathcal{A}_u(B_Y)$ *is lush.*
(2) $\mathcal{A}_{w^*u}(B_{Y^*}, X)$ *is lush.*
(3) $\mathcal{A}_{wu}(B_Y, X)$ *is lush.*

We finally study the applications of the Urysohn type lemmas to the *AHSP* of A^X. A Banach space X is said to have the *AHSP* if for every $\epsilon > 0$ there exist $\gamma(\epsilon) > 0$ and $\eta(\epsilon) > 0$ with $\lim_{\epsilon\to 0^+} \gamma(\epsilon) = 0$ such that for every sequence $(x_k)_{k=1}^{\infty} \subset B_X$ and for every convex series $\sum_{k=1}^{\infty} \alpha_k$ satisfying

$$\left\|\sum_{k=1}^{\infty} \alpha_k x_k\right\| > 1-\eta(\epsilon)$$

there exist a subset $A \subset \mathbb{N}$, a subset $\{z_k : k \in A\} \subset S_X$ and $x^* \in S_{X^*}$ such that

(i) $\sum_{k\in A} \alpha_k > 1-\gamma(\epsilon)$
(ii) $\|z_k - x_k\| < \epsilon$ for all $k \in A$, and
(iii) $x^*(z_k) = 1$ for all $k \in A$.

It is easy to check that the *AHSP* holds if it is satisfied just for every finite convex combination instead of every infinite convex series. This property was introduced in [1] to characterize a Banach space X such that the pair (ℓ_1, X) has the Bishop–Phelps–Bollobás property for operators. The following Banach spaces were shown to have the *AHSP* (see [1,8]): (a) a finite dimensional space, (b) a real or complex space $L_1(\mu)$ for a σ-finite measure μ, (c) a uniformly convex space, (d) a lush space, and (see [2]) (e) the space of continuous mappings $C(K, X)$ defined on a compact Hausdorff topological space K with values in a uniformly convex Banach space X. Since every lush space has the *AHSP*, Theorem 2.12 implies the following.

Corollary 2.14. *Suppose that A is a uniform algebra.*

(1) *If X is lush, then A^X has the* AHSP. *In particular, A has the* AHSP.
(2) *If A is unital and A^X is lush, then X has the* AHSP.

Now we generalize this corollary.

Theorem 2.15. *Suppose that A is a uniform algebra.*

(1) *If X has the* AHSP, *then A^X has the* AHSP.
(2) *If A is unital and A^X has the* AHSP, *then X has the* AHSP.

Proof. (1) Assume that X has the *AHSP* with the positive numbers $\eta(\epsilon)$ and $\gamma(\epsilon)$ for $\epsilon > 0$. Let $(f_k)_{k=1}^{n} \subset B_{A^X}$ and a finite convex series $\sum_{k=1}^{n} \alpha_k$ satisfying

$$\left\|\sum_{k=1}^{n} \alpha_k f_k\right\| > 1-\eta(\epsilon/3).$$

Choose $t_1 \in \Gamma_0$ so that

$$\left\|\sum_{k=1}^{n} \alpha_k f_k(t_1)\right\| > 1-\eta(\epsilon/3).$$

Find $A \subset \{1, \ldots, n\}$ such that there exist $\{z_k : k \in A\} \subset S_X$ and $z^* \in S_{z^*}$ such that $\sum_{k\in A} \alpha_k > 1-\gamma(\epsilon/3)$, $\|z_k - f_k(t_1)\| < \epsilon/3$ for all $k \in A$ and $z^*(z_k) = 1$ for all $k \in A$. Consider an open set

$$U = \bigcap_{k\in A} \big\{t \in K : \|f_k(t) - f_k(t_1)\| < \epsilon\big\}.$$

From Lemmas 1.1 and 1.2 it follows that given $\epsilon/6 > 0$ there exist $\phi \in A$ and $t_2 \in \Gamma_0 \cap U$ such that $\phi(t_2) = \|\phi\|_\infty = 1$, $|\phi(t)| < \epsilon/6$ for every $t \in K \setminus U$ and

$$|\phi(t)| + \left(1-\frac{\epsilon}{6}\right)|1-\phi(t)| \le 1, \quad \text{for all } t \in K.$$

Define for $k \in A$

$$g_k = \phi z_k + \left(1 - \frac{\epsilon}{6}\right)(\mathbf{1} - \phi) f_k \in A^X.$$

Then,

$$\|g_k(t)\| \le |\phi(t)| + \left(1 - \frac{\epsilon}{6}\right)|1 - \phi(t)| \le 1 \quad \text{and} \quad g_k(t_2) = z_k.$$

We get that for $t \in U$

$$\begin{aligned}\big\|g_k(t) - f_k(t)\big\| &= \big\|\big(z_k - f_k(t)\big)\phi(t) - \big(1 - \phi(t)\big)f_k(t)\big\| \\ &\le \big\|z_k - f_k(t)\big\| + \left\|\frac{\epsilon}{6}\big(1 - \phi(t)\big)f_k(t)\right\| \\ &< \big\|z_k - f_k(t_1)\big\| + \big\|f_k(t_1) - f_k(t)\big\| + \frac{\epsilon}{3} \\ &\le \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,\end{aligned}$$

and for $t \in K \setminus U$

$$\begin{aligned}\big\|g_k(t) - f_k(t)\big\| &= \left\|\big(z_k - f_k(t)\big)\phi(t) - \frac{\epsilon}{6}\big(1 - \phi(t)\big)f_k(t)\right\| \\ &\le |\phi(t)|\left\|z_k - f_k(t)\right\| + \left\|\frac{\epsilon}{6}\big(1 - \phi(t)\big)f_k(t)\right\| \\ &\le \epsilon.\end{aligned}$$

The fact that $\left(z^* \circ \delta_{t_2}\right)(g_k) = 1$ for every $k \in A$ completes the proof.

(2) Assume that A^X has the *AHSP* with the positive numbers $\eta(\epsilon)$ and $\gamma(\epsilon)$ for $\epsilon > 0$. Given a sequence $(x_k)_{k=1}^{\infty} \subset B_X$ and a convex series $\sum_{k=1}^{\infty} \alpha_k$, assume that

$$\left\|\sum_{k=1}^{\infty} \alpha_k x_k\right\| > 1 - \eta(\epsilon).$$

Since A is unital, for each $k \in \mathbf{N}$ we define $f_k \in B_{A^X}$ by $f_k(t) = x_k$ for all $t \in K$. We can get

$$\left\|\sum_{k=1}^{\infty} \alpha_k f_k\right\| > 1 - \eta(\epsilon).$$

By the assumption there exist $A \subset \mathbb{N}$, $\{g_k : k \in A\} \subset S_{A^X}$ and $\phi \in S_{(A^X)^*}$ such that $\sum_{k\in A} \alpha_k > 1 - \gamma(\epsilon)$, $\|g_k - f_k\| < \epsilon$ for all $k \in A$, and $\phi(g_k) = 1$ for all $k \in A$. From these it follows that $\| \sum_{k\in A} \alpha_k g_k\| = \sum_{k\in A} \alpha_k$. Choose $t_0 \in K$ so that

$$\left\|\sum_{k\in A} \alpha_k g_k(t_0)\right\| = \sum_{k\in A} \alpha_k.$$

Choose also $x^* \in S_X$ so that $x^*\left(\sum_{k\in A} \alpha_k g_k(t_0)\right) = \sum_{k\in A} \alpha_k$. Put $z_k = g_k(t_0)$ for every $k \in A$. Clearly $\|x_k - z_k\| < \epsilon$ and $x^*(z_k) = 1$ for every $k \in A$, hence X has the *AHSP*. □

For a complex Banach space X, it follows from Theorem 2.15 that $C(K, X)$ has the *AHSP* if and only if X has the *AHSP*. In case of a real Banach space X, it was shown in [8, Theorem 11] that if $C(K, X)$ has the *AHSP*, then X has the *AHSP*. The other implication can be easily proved by modifying the proof of Theorem 2.15(1) with the classical Urysohn lemma. Indeed, assume that X has the *AHSP* with the positive numbers $\eta(\epsilon)$ and $\gamma(\epsilon)$ for $\epsilon > 0$. Let $(f_k)_{k=1}^{n} \subset B_{C(K,X)}$ and a finite convex series $\sum_{k=1}^{n} \alpha_k$ satisfying

$$\left\|\sum_{k=1}^{n} \alpha_k f_k\right\| > 1 - \eta(\epsilon).$$

Choose $t_0 \in K$ so that

$$\left\|\sum_{k=1}^{n} \alpha_k f_k(t_0)\right\| > 1 - \eta(\epsilon).$$

We next take an open neighborhood U of t_0 such that for every $t \in U$

$$\left\| \sum_{k=1}^{n} \alpha_k f_k(t) \right\| > 1 - \eta(\epsilon) \quad \text{and} \quad \|f_k(t) - f_k(t_0)\| < \epsilon$$

for every $k = 1, \ldots, n$. Since X has the *AHSP*, we can find $A \subset \mathbb{N}$ such that there exist $\{z_k : k \in A\} \subset S_X$ and $z^* \in S_{Z^*}$ satisfying: $\sum_{k\in A} \alpha_k > 1 - \gamma(\epsilon)$, $\|z_k - f_k(t_0)\| < \epsilon$ for all $k \in A$ and $z^*(z_k) = 1$ for all $k \in A$. From the classical Urysohn lemma, there exists a non-negative function $\phi \in C(K)$ such that $\phi(t_0) = \|\phi\|_\infty = 1$, and $|\phi(t)| = 0$ for every $t \in K \setminus U$. Define

$$g_k = \phi z_k + (1-\epsilon)(\mathbf{1} - \phi) f_k \in C(K, X).$$

Then, $|g_k(t)| \le |\phi(t)| + (1-\epsilon)|1 - \phi(t)| \le \phi(t) + 1 - \phi(t) = 1$ and $g_k(t_0) = z_k$. We also can see that for $t \in U$

$$\begin{aligned} \|g_k(t) - f_k(t)\| &= \big\|\big(z_k - f_k(t)\big)\phi(t) - \epsilon\big(1 - \phi(t)\big)f_k(t)\big\| \\ &\le \|z_k - f_k(t)\| + 2\epsilon < \|z_k - f_k(t_0)\| + \|f_k(t_0) - f_k(t)\| + 2\epsilon \\ &\le 4\epsilon, \end{aligned}$$

and for $t \in K \setminus U$

$$\begin{aligned} |g_k(t) - f_k(t)| &= \big\|\big(z_k - f_k(t)\big)\phi(t) - \epsilon\big(1 - \phi(t)\big)f_k(t)\big\| \\ &\le \epsilon, \end{aligned}$$

and $\left(z^* \circ \delta_{t_0}\right)(g_k) = 1$ for every $k \in A$.

Remark 2.16. Theorems 2.1, 2.7, 2.11, 2.12 and 2.15 also hold for every subspace $B \subset A^X$ which satisfies $A \bigotimes X \subset B$ and $fg \in B$ for every $f \in A$ and $g \in B$ proven in an analogous way.

Corollary 2.17. *If X has the* AHSP, *then for any Banach space Y*

(1) $\mathcal{A}_u(B_Y)$ *has the* AHSP.
(2) $\mathcal{A}_{w^*u}(B_{Y^*}, X)$ *has the* AHSP.
(3) $\mathcal{A}_{wu}(B_Y, X)$ *has the* AHSP.

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