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The Fourier finite element method for the corner singularity expansion of the Heat equation

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ABSTRACT

Near the non-convex vertex the solution of the Heat equation is of the form $u = (c \star \mathcal{E})\chi r^{\pi/\omega} \sin(\frac{\pi\theta}{\omega}) + w$, $w \in L^2(\mathbb{R}^+; H^2)$, where c is the stress intensity function of the time variable t, $\star$ the convolution, $\mathcal{E}(\mathbf{x}, t) = re^{-r^2/4t}/2\sqrt{\pi t^3}$, χ a cutoff function and ω the opening angle of the vertex. In this paper we use the Fourier finite element method for approximating the stress intensity function c and the regular part w, and derive the error estimates depending on the regularities of c and w. We give some numerical examples, confirming the derived convergence rates.

1. Introduction

The purpose of this paper is to apply the Fourier finite element method (FFEM) to the corner singularity expansion for the Heat equation on non-convex polygonal domains, to show unique existence and error estimates, and to confirm the derived convergence rates by numerical experiments. The FFEM combines the Fourier method with the finite element method. This provides some advantages. The approximate solution of the Heat equation on polygonal domains or the boundary value problem on axisymmetric domains can be reduced to the approximate solution of a finite set of boundary value problems in two dimensions and can be solved in parallel. Also, the approximate stress intensity function can be calculated by truncated Fourier series, with coefficients of singular functions in two dimensions (see [1]).

The FFEM is based on the method given in Ref. [2], where it was applied to a general second-order elliptic Dirichlet boundary value problem on axisymmetric domains in $\mathbb{R}^3$. In [1] it was also applied to the Dirichlet problem of the Poisson equation in axisymmetric domains with reentrant edges and in [3] the interface problem of the Poisson-like equation in axisymmetric domains with edges. In [4,5] the Fourier singular complement method was introduced and analyzed in such axisymmetric domains. In [6] a combination of the FFEM with the Nitsche finite element method was applied to the Dirichlet problem of the Poisson equation in 3D axisymmetric domains with non-axisymmetric data. In particular, compared with this paper, the analysis given in [7] is similar in applying the FFEM to the edge singularity expansion of the Poisson problem but, in this paper, the stress intensity factor of the corner singularity for the Poisson problem with parameter is differently formulated (see Remark 1.3).

This paper provides a FFEM to overcome corner singularities for the Heat equation with (nearly) optimal order convergence. For the optimality we try to find the approximation of the regular part in the corner singularity expansion before computing the stress intensity function of the singular solution (cf. [7]). Even though our approach needs the knowledge of the exact forms of corner singularities, its advantage is that there is no procedure refining triangulations near the non-convex corner, compared with Ref. [8].

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Handling singularity caused by singular boundaries is a very important issue in mechanics and numerical computations because, as expected, the corners and edges of boundary may result in mechanically singular forces and numerically unstable phenomena. The finite element methods to overcome the corner singularities of the second order elliptic boundary value problem were investigated in [9–15]. Even the singularities of the parabolic problem were studied so some numerical methods on domains with corners were analyzed in [8,16–18]. In this paper we also consider the initial and boundary value problems for the Heat equation:

$$\begin{aligned} &\partial_t u - \Delta u = f \quad \text{in } \Omega \times \mathbb{R}^+, \\ &u = 0 \quad \text{on } \Gamma \times \mathbb{R}^+, \\ &u(\cdot, 0) = 0 \quad \text{in } \Omega, \end{aligned} \tag{1.1}$$

where u is the unknown function and f a given function; $\Omega \subset \mathbb{R}^2$ is a non-convex polygon with the boundary $\Gamma := \partial\Omega$; $\mathbb{R}^+ := (0, \infty)$ is the positive real line.

It is assumed, for simplicity, that Ω has only one non-convex vertex P placed at the origin. Let $\omega = \omega_2 - \omega_1 > \pi$ be the opening angle of P where ω_i are the numbers in $\omega_1 < \omega_2 < \omega_1 + 2\pi$. Set $\alpha = \pi/\omega$. Denote by $r > 0$ the radial coordinate and $\theta \in (\omega_1, \omega_2)$ the angular coordinate. The corresponding corner singularity ϕ and its dual function ψ are given by

$$\phi = \chi_1 r^{\alpha} \sin[\alpha(\theta - \omega_1)], \qquad \psi = \chi_2 r^{-\alpha} \sin[\alpha(\theta - \omega_1)], \tag{1.2}$$

where $\chi_i \in C^{\infty}(\mathbb{R}^2)$ are the cutoff functions defined by

$$\chi_j = 1 \quad \text{for } r \le d_j \text{ and } 0 \text{ for } r \ge d_{j+1}, \tag{1.3}$$

where d_j are numbers with $0 < d_1 < d_2 < d_3 \ll 1$.

The spaces and norms used in this paper are as follows. For real s, H^s means the usual fractional order Sobolev space with norm $\|v\|_s$ (see [19,20]). We write $\mathrm{L}^2 = \mathrm{H}^0$ with norm $\|v\|_0 = (\int_\Omega |v|^2 dx)^{1/2}$, $\mathrm{H}^1_0 := \{v \in \mathrm{H}^1 : v|_\Gamma = 0\}$ and $\mathrm{H}^s_0 = \mathrm{H}^s \cap \mathrm{H}^1_0$. Also H^{-s} means the dual space of H^s_0 with norm

$$\|f\|_{-s} := \sup_{0 \ne v \in \mathrm{H}^s_0} \langle f, v\rangle / \|v\|_s, \tag{1.4}$$

where $\langle\, , \,\rangle$ denotes the duality pairing. The function $u(\mathbf{x}, t)$ is considered as a mapping $u : \mathbb{R}^+ \mapsto X$ defined by $[u(t)](\mathbf{x}) := u(\mathbf{x}, t)$ for $\mathbf{x} \in \Omega$ and $t \in \mathbb{R}^+$, where X is a Banach space with norm $\|\cdot\|$. Let $\mathrm{L}^2(\mathbb{R}^+; X)$ be the set of all measurable functions with

$$\|u\|_{\mathrm{L}^2(\mathbb{R}^+;X)} := \left(\int_0^\infty \|u(t)\|^2 dt\right)^{1/2}.$$

Throughout this paper C denotes a generic positive constant, for instance, $C = C(\Omega, \ldots)$, depending on Ω and so on.

The solution of the elliptic boundary value problems on a polygonal domain, for instance, the Poisson problem: $-\Delta u = f$ in Ω and $u = 0$ on Γ can be written in the following form near the non-convex vertex (see [20–23]):

$$u = \mathfrak{C}\phi + w, \quad w \in \mathrm{H}^2$$

with the regularity estimate: $\|w\|_2 + |\mathfrak{C}| \le C\|f\|_0$ for a constant C. In the numerical analysis a main issue is how the (optimally) convergent numerical solutions for the pair $[\mathfrak{C}, w]$ can be constructed. Such investigation can be found in the following references: [11–15]. In [11] the extraction formula for the coefficient $\mathfrak{C}$ is presented, based on the dual singular function method, and the error estimates are derived: $|\mathfrak{C} - \mathfrak{C}_h| + \|w - w_h\|_0 = O(h^{1+\alpha-\epsilon})$ for $0 < \epsilon \ll 1$ and $\|w - w_h\|_1 = O(h)$ where w_h and $\mathfrak{C}_h$ are the approximations of w and $\mathfrak{C}$ respectively. In [13] the multi-grid methods for the computation of singular solutions and stress intensity factors are studied and the error estimates are derived: $|\mathfrak{C} - \mathfrak{C}_h| = O(h^{1+\alpha-\epsilon})$, $\|w - w_h\|_1 = O(h)$. In [14,15] the extraction formula given in [11] is modified by the expression containing only the regular part w and the discrete variable w_h is computed by using the Sherman–Morrison formula and also the error estimates: $|\mathfrak{C} - \mathfrak{C}_h| + \|w - w_h\|_0 = O(h^{1+\alpha-\epsilon})$, $\|w - w_h\|_1 = O(h)$ are shown. Furthermore, some noticeable works on the finite element methods for elliptic boundary value problems on domains with cusps can be found in [24,25].

On the other hand such numerical analysis for the corner singularity decomposition of the Heat equation has not been given yet. A direct numerical approach to the solution itself can be found in the references: [8,16–18]. In [8] the authors show that the approximation u_h for the semidiscrete formulation of the problem (1.1) satisfies the error estimates: $\|u(t) - u_h(t)\|_0 = O(h^{2\alpha})$, $\|\nabla(u(t) - u_h(t))\|_0 = O(h^{\alpha})$ and also the optimal order convergence rates can be restored by systematically refining triangulations toward the non-convex corner.

Unlike the corner singularity expansion of the Poisson problem the corner singularity of a non-convex vertex for the time-dependent problem (1.1) corresponds to each time $t > 0$ and is of the form (Theorem 1.1)

$$u = (\mathcal{E} \star c)\phi + w, \tag{1.5}$$

where $\star$ is the convolution in time, ϕ is the corner singularity in (1.2) and w is the smoother part. We here state the regularity result for the Heat equation (1.1) on the non-convex polygon Ω (see [26, Theorem 2.2]).

Theorem 1.1. *If $f \in \mathrm{L}^2(\mathbb{R}^+; \mathrm{H}^{-1})$, there is a solution $u \in \mathrm{L}^2(\mathbb{R}^+; \mathrm{H}^1_0)$ of (1.1), satisfying*

$$\operatorname*{ess\,sup}_{0\le t<\infty} \|u(t)\|_0 + \|u\|_{\mathrm{L}^2(\mathbb{R}^+,\mathrm{H}^1)} + \|\partial_t u\|_{\mathrm{L}^2(\mathbb{R}^+;\mathrm{H}^{-1})} \le C\|f\|_{\mathrm{L}^2(\mathbb{R}^+;\mathrm{H}^{-1})}.$$

On the other hand, if $f \in \mathrm{L}^2(\mathbb{R}^+; \mathrm{L}^2)$, the solution u is written by

$$u(t) = (\mathcal{E} \star c)(t)\phi + w(t), \quad w(t) := u(t) - (\mathcal{E} \star c)(t)\phi, \tag{1.6a}$$

$$c(t) = \frac{1}{2\pi i}\int_\gamma \langle \Lambda^\lambda; (\lambda \mathrm{I} - \partial_t)^{-1} f(t)\rangle d\lambda, \qquad \mathcal{E}(\mathbf{x}, t) = re^{-r^2/4t}/2\sqrt{\pi t^3}, \tag{1.6b}$$

where γ is a vertical axis satisfying $\mathrm{Re}\,\lambda < 0$ for $\lambda \in \gamma$, and Λ^λ is a continuous linear functional on H^{s-2} defined by (1.17). Furthermore the pair $[w, c] \in \mathrm{L}^2(\mathbb{R}^+; \mathrm{H}^2) \times \mathrm{H}^{(1-\alpha)/2}(\mathbb{R}^+)$ and satisfies

$$\operatorname*{ess\,sup}_{0\le t<\infty} \|w(t)\|_1 + \|w\|_{\mathrm{L}^2(\mathbb{R}^+;\mathrm{H}^2)} + \|\partial_t w\|_{\mathrm{L}^2(\mathbb{R}^+;\mathrm{L}^2)} + \|c\|_{(1-\alpha)/2,\mathbb{R}^+} \le C\|f\|_{\mathrm{L}^2(\mathbb{R}^+;\mathrm{L}^2)}.$$

We shall be concerned with the periodic extension in time t of the solution u in (1.6a) by the Fourier series expansion. Let T be a positive number. Let $\Omega_T = \Omega \times (0, T)$ (see Fig. 1). The Fourier series expansion of $v \in \mathrm{L}^2(\Omega_\mathrm{T})$ is given by

$$v(\mathbf{x}, t) = \sum_{k=-\infty}^{\infty} V_k(\mathbf{x})e^{\lambda_k t}, \quad \lambda_k = 2\pi ki/\mathrm{T}, \tag{1.7}$$

where V_k is the kth Fourier coefficient (Ref. [27, Section 2.3]) of v with respect to t, defined by

$$V_k(\mathbf{x}) = \frac{1}{\mathrm{T}}\int_0^{\mathrm{T}} v(\mathbf{x}, t)e^{-\lambda_k t}dt. \tag{1.8}$$

Since the solution u is square integrable on Ω_T, the Fourier series converges to the solution at almost every point in Ω_T and works for a general solution u on Ω_T. Letting $\mathfrak{C}_k$ and W_k be the Fourier coefficients of c and w, the Fourier series expansion of (1.5) is given by

$$u = \sum_{k=-\infty}^{\infty} U_k e^{\lambda_k t}, \quad U_k = \mathfrak{C}_k e^{-r\sqrt{\lambda_k}}\phi + W_k, \tag{1.9}$$

which follows by

$$\int_0^{\mathrm{T}}\int_0^\infty \mathcal{E}(s)c(t-s)ds\, e^{-\lambda_k t}dt = \int_0^\infty \mathcal{E}(s)e^{-\lambda_k s}ds \int_0^{\mathrm{T}} c(t)e^{-\lambda_k t}dt = e^{-r\sqrt{\lambda_k}}\,\mathfrak{C}_k\mathrm{T}.$$

To approximate the decomposition (1.9) we follow the following steps:

(a) Find $U \in \mathrm{H}^1_0$ such that $-\Delta U + \lambda U = F$ in Ω.
(b) Solve a (generalized) boundary value problem for $W := U - \mathfrak{C}e^{-r\sqrt{\lambda}}\phi$.
(c) Construct the approximation $[\mathfrak{C}_h, W_h]$ of $[\mathfrak{C}, W]$ for the mesh-size h.
(d) Find the approximation $[\mathfrak{C}_{k,h}, W_{k,h}]$ corresponding to $\lambda = \lambda_k$ and define a finite approximation for the sum (1.9):

$$u_h^N := \sum_{k=-N}^{N} U_{k,h}e^{\lambda_k t}, \quad U_{k,h} := \mathfrak{C}_{k,h}e^{-r\sqrt{\lambda_k}}\phi + W_{k,h}. \tag{1.10}$$

(e) With $u^N := u - \sum_{|k|>N} U_k e^{\lambda_k t}$, we analyze the error defined by

$$u^N - u_h^N = \sum_{k=-N}^{N}(\mathfrak{C}_k - \mathfrak{C}_{k,h})e^{-r\sqrt{\lambda_k}+\lambda_k t}\phi + \sum_{k=-N}^{N}(W_k - W_{k,h})e^{\lambda_k t}.$$

In order to follow above procedure we insert the Fourier series of u and f into (1.1), equate each coefficient of $e^{\lambda_k t}$ and get the following equation for the Fourier coefficients:

$$-\Delta U_k + \lambda_k U_k = F_k \quad \text{in } \Omega,\ U_k = 0 \text{ on } \Gamma, \tag{1.11}$$

where U_k and F_k are the Fourier coefficients of u and f, respectively.

The problem (1.11) will be solved by investigating the following Poisson problem with parameter:

$$-\Delta U + \lambda U = F \quad \text{in } \Omega,\ U = 0 \text{ on } \Gamma, \tag{1.12}$$

where λ is a complex number with $\mathrm{Re}\,\lambda \ge 0$. We here state a regularity result for (1.12). The proof can be found in the Refs. [28, Theorem 2.1] and [29].

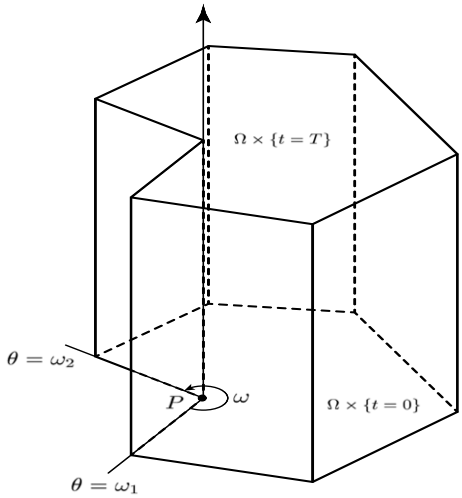


Fig. 1. The region Ω_T.

Theorem 1.2. *If $F \in \mathrm{H}^{-1}$, there exists a unique solution $U \in \mathrm{H}^1_0$ of (1.12), satisfying $\|U\|_1 + |\lambda|^{1/2}\|U\|_0 \le C\|F\|_{-1}$ for a constant C. On the other hand, there exist a number $\mathfrak{C}$ and a function $W \in \mathrm{H}^2$ such that if $F \in \mathrm{L}^2$, the solution U can be written by*

$$U = \mathfrak{C}e^{-r\sqrt{\lambda}}\phi + W, \tag{1.13}$$

where $\phi = \chi_1 r^\alpha \sin[\alpha(\theta - \omega_1)]$. Furthermore there is a constant C such that the following regularity holds:

$$\begin{aligned} &\|W\|_2 + (1+|\lambda|)^{1/2}\|W\|_1 + (1+|\lambda|)\|W\|_0 \le C\|F\|_0, \\ &(1+|\lambda|)^{(1-\alpha)/2}|\mathfrak{C}| \le C\|F\|_0. \end{aligned} \tag{1.14}$$

The coefficient $\mathfrak{C}$ of (1.13) is formulated as follows: For simplicity we set

$$\begin{aligned} &\phi_\lambda = e^{-r\sqrt{\lambda}}\phi, \qquad \psi_\lambda = e^{-r\sqrt{\overline{\lambda}}}\psi, \\ &\phi^*_\lambda = (\Delta - \lambda \mathrm{I})\phi_\lambda, \qquad \psi^*_\lambda = (\Delta - \overline{\lambda}\mathrm{I})\psi_\lambda, \end{aligned} \tag{1.15}$$

where $\overline{\lambda}$ denotes the complex conjugate of λ and I is the identity operator. Using $\varphi \in \mathrm{H}^1$ satisfying $(-\Delta + \lambda)\varphi = \psi^*_\lambda \in \mathrm{H}^{-1}$ with $\varphi|_\Gamma = -\psi_\lambda|_\Gamma$ and $F = -(\Delta - \lambda\mathrm{I})U$, we write the coefficient $\mathfrak{C}$ of (1.13) by

$$\mathfrak{C} = \Lambda^\lambda(F) := \frac{1}{\pi}\int_\Omega F\overline{(\varphi + \psi_\lambda)}d\mathbf{x} \tag{1.16}$$

$$= \frac{1}{\pi}\int_\Omega U\overline{\psi^*_\lambda} + F\,\overline{\psi_\lambda}d\mathbf{x}. \tag{1.17}$$

For a detailed derivation for (1.16) one may refer to [28, Section 2]. A similar formulation like (1.17) can be found in [14, (1.8)] and [20, pp. 399].

Remark 1.3. In this paper we shall use the formula (1.17) for $\mathfrak{C}$ instead of (1.16). A main reason is that the expression (1.17) enables us to derive a better L^2-error estimate for the regular part W of (1.13): $O(h^{2-\alpha-\epsilon})$ with $0 < \epsilon \ll 1-\alpha$ (see (1.25a)). Meanwhile, in [7, Section 3] the formula (1.16) is used and the derived L^2-error estimate for the regular part W is $O(h^{1-\epsilon})$ (see [7, Theorem 3.3]).

Using $U = \mathfrak{C}\phi_\lambda + W$ of (1.13), we split the formula (1.17) for $\mathfrak{C}$ by

$$\mathfrak{C} = \mathfrak{C}_1(W) + \mathfrak{C}_2(F), \tag{1.18}$$

where

$$\begin{aligned} &\mathfrak{C}_1(W) = \frac{1}{\pi\gamma(\lambda)}\int_\Omega W\overline{\psi^*_\lambda}d\mathbf{x}, \qquad \mathfrak{C}_2(F) = \frac{1}{\pi\gamma(\lambda)}\int_\Omega F\overline{\psi_\lambda}d\mathbf{x}, \\ &\gamma(\lambda) = 1 - \frac{1}{\pi}\int_\Omega \phi_\lambda\,\overline{\psi^*_\lambda}d\mathbf{x}. \end{aligned} \tag{1.19}$$

Inserting $U = \mathfrak{C}\phi_\lambda + W$ into (1.12) and using (1.18), one has a generalized boundary value problem for the remainder W:

$$\begin{aligned} -\Delta W + \lambda W - \mathfrak{C}_1(W)\phi_\lambda^* &= F_\lambda \quad \text{in } \Omega, \\ W &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{1.20}$$

where $F_\lambda := \mathfrak{C}_2(F)\phi_\lambda^* + F$. We shall find the weak solution $W \in \mathrm{H}_0^1$ of (1.20), satisfying

$$a_\lambda(W, V) = \langle F_\lambda, V\rangle, \quad \forall V \in \mathrm{H}_0^1, \tag{1.21}$$

where

$$a_\lambda(W, V) = \langle \nabla W, \nabla V\rangle + \lambda\langle W, V\rangle - \mathfrak{C}_1(W)\langle \phi_\lambda^*, V\rangle,$$

$$\langle F_\lambda, V\rangle = \int_\Omega F_\lambda \overline{V} d\mathbf{x}.$$

Set $\mathcal{V} = \mathrm{H}_0^1$. Let $\mathcal{V}_h$ be a finite dimensional subspace of $\mathcal{V}$ for the mesh-size h. Then the approximate problem is to find $W_h \in \mathcal{V}_h$ such that

$$a_\lambda(W_h, V_h) = \langle F_\lambda, V_h\rangle, \quad \forall V_h \in \mathcal{V}_h. \tag{1.22}$$

With W_h of (1.22) we define the approximation of the coefficient $\mathfrak{C}$ in (1.13) by

$$\mathfrak{C}_h = \mathfrak{C}_1(W_h) + \mathfrak{C}_2(F). \tag{1.23}$$

Recall from (1.14) that a least regularity space for F is assumed to be L^2 so that the stress intensity factor $\mathfrak{C}$ can be well defined and the H^2-regularity for the remainder W can be obtained.

We here give unique existences for W_h and $\mathfrak{C}_h$ and the error estimates, which are shown in Theorems 3.3–3.4.

Theorem 1.4. *Let λ be a parameter with $\mathrm{Re}\,\lambda \geq 0$. If $F \in \mathrm{L}^2$, there exists a unique solution $W_h \in \mathcal{V}_h$ of (1.22). Also the approximation $\mathfrak{C}_h$ is defined by (1.23) and the following a priori estimates hold:*

$$(1 + |\lambda|)^{1/2}\|\nabla W_h\|_0 + (1 + |\lambda|)\|W_h\|_0 \leq C\|F\|_0, \tag{1.24a}$$

$$(1 + |\lambda|)^{(1-\alpha)/2}|\mathfrak{C}_h| \leq C\|F\|_0. \tag{1.24b}$$

Furthermore, there is a constant C independent of h such that, for a small number $0 < \epsilon \ll 1 - \alpha$,

$$\|W - W_h\|_0 \leq Ch^{2-\alpha-\epsilon}\|F\|_0, \tag{1.25a}$$

$$\|\nabla(W - W_h)\|_0 + (1 + |\lambda|)^{1/2}\|W - W_h\|_0 \leq Ch\|F\|_0, \tag{1.25b}$$

$$|\mathfrak{C} - \mathfrak{C}_h| \leq C(1 + |\lambda|)^{\frac{\alpha}{2}} h\|F\|_0. \tag{1.25c}$$

In particular it is noted that if the parameter λ is set to be zero in the problems (1.21) and (1.22), one has $\|W - W_h\|_0 \leq Ch^{1+\alpha-\epsilon}\|F\|_0$ and $|\mathfrak{C} - \mathfrak{C}_h| \leq Ch^{1+\alpha-\epsilon}\|F\|_0$ for any ϵ with $0 < \epsilon \ll \alpha$.

Remark 1.5. As mentioned in Remark 1.3, it is noted that the error estimate (1.25a) gives a better L^2-error estimate for the regular part W than that given in [7, Theorem 3.3]. Since our approach is firstly to find the approximation of $W \in \mathrm{H}^2$, the numerical tests in Section 4.1 show a super-convergence for the approximation of W with the rate 2 in L^2-error (cf. [13,15]) but it is hard to derive the best rate 2 in theory. Such difficulty comes from a lack of the regularity $\mathrm{H}^{2-\alpha-\epsilon}$, shown in Lemma 3.1, for the dual problem (3.1) with $g = W - W_h$ (see Theorem 3.4).

We here recall the Fourier series $u = \sum_{k=-\infty}^{\infty} U_k e^{\lambda_k t}$. If $u = (\mathcal{E} \star c)\phi + w$, the Fourier coefficient can be written by $U_k = \mathfrak{C}_k e^{-r\sqrt{\lambda_k}}\phi + W_k$ and

$$c = \sum_{k=-\infty}^{\infty} \mathfrak{C}_k e^{\lambda_k t}, \qquad w = \sum_{k=-\infty}^{\infty} W_k e^{\lambda_k t}. \tag{1.26}$$

For $\lambda = \lambda_k$, the coefficient $W_k \in \mathrm{H}_0^1$ is the weak solution satisfying $a_\lambda(W_k, V) = \langle F_\lambda, V\rangle, \forall V \in \mathrm{H}_0^1$ and also $\mathfrak{C}_k = \mathfrak{C}_1(W_k) + \mathfrak{C}_2(F_k)$. Let $W_{k,h}$ be the discrete solution of (1.22) with $\lambda = \lambda_k$ and $\mathfrak{C}_{k,h}$ the approximate coefficient of (1.23). Here we define the approximation $[w_h^N, c_h^N]$ of the truncation $[w^N, c^N]$ by

$$w_h^N = \sum_{k=-N}^{N} W_{k,h} e^{\lambda_k t}, \qquad c_h^N = \sum_{k=-N}^{N} \mathfrak{C}_{k,h} e^{\lambda_k t}. \tag{1.27}$$

To measure the periodic extension of the solution we consider the following spaces: Let $\mathcal{X}$ be a Banach space with norm $\|\cdot\|_\mathcal{X}$ on Ω. We define $\mathrm{L}^2(0, \mathrm{T}; \mathcal{X}) = \{v : \|v\|_{\mathrm{L}^2(0,\mathrm{T};\mathcal{X})} < \infty\}$ with norm $\|v\|_{\mathrm{L}^2(0,\mathrm{T};\mathcal{X})} := \mathrm{T}^{1/2}(\sum_{k=-\infty}^{\infty} \|V_k\|_\mathcal{X}^2)^{1/2}$ and for real $s > 0$, $\mathrm{H}^s(0, \mathrm{T}; \mathcal{X}) = \{v \in \mathrm{L}^2(0, \mathrm{T}; \mathcal{X}) : \|v\|_{\mathrm{H}^s(0,\mathrm{T};\mathcal{X})} < \infty\}$ with norm

$$\|v\|_{\mathrm{H}^s(0,\mathrm{T};\mathcal{X})} := \mathrm{T}^{1/2}\Big(\sum_{k=-\infty}^{\infty} (1 + |\lambda_k|)^{2s}\|V_k\|_\mathcal{X}^2\Big)^{1/2}.$$

Also, we consider $L^2(0,T) = \{c : \|c\|_{L^2(0,T)} < \infty\}$ with norm $\|c\|_{L^2(0,T)} := T^{1/2}(\sum_{k=-\infty}^{\infty} |\mathfrak{C}_k|^2)^{1/2}$ and $H^s(0,T) = \{c \in L^2(0,T) : \|c\|_{H^s(0,T)} < \infty\}$ with norm

$$\|c\|_{H^s(0,T)} := T^{1/2}\Big(\sum_{k=-\infty}^{\infty} (1+|\lambda_k|)^{2s}|\mathfrak{C}_k|^2\Big)^{1/2}.$$

We here give the a priori estimate for the approximation of (1.27) and the error estimates, which are shown in Section 3.

Theorem 1.6. *If $f \in L^2(0,T;L^2)$ then the approximation $[w_h^N, c_h^N]$ of (1.27) satisfies $\|w_h^N\|_{H^{1/2}(0,T;H^1)} + \|w_h^N\|_{H^1(0,T;L^2)} + \|c_h^N\|_{H^{(1-\alpha)/2}(0,T)} \le C\|f\|_{L^2(0,T;L^2)}$ for a constant C. On the other hand, let $[w, c]$ be of the form (1.26). Then there exists a constant C independent of h and N such that, for $0 < \epsilon \ll 1-\alpha$,*

$$\text{(i) } \|w - w_h^N\|_{L^2(0,T;L^2)} \le C(h^{2-\alpha-\epsilon} + N^{-1})\|f\|_{L^2(0,T;L^2)}, \tag{1.28a}$$

$$\text{(ii) } \|w - w_h^N\|_{L^2(0,T;H^1)} \le C(h + N^{-1/2})\|f\|_{L^2(0,T;L^2)}, \tag{1.28b}$$

$$\text{(iii) } \|w - w_h^N\|_{H^{1/2}(0,T;L^2)} \le C(h + N^{-1/2})\|f\|_{L^2(0,T;L^2)}. \tag{1.28c}$$

Additionally, if $f \in H^{1/2}(0,T;L^2)$ then

$$\text{(iv) } \|w - w_h^N\|_{H^{1/2}(0,T;H^1)} \le C(h + N^{-1/2})\|f\|_{H^{1/2}(0,T;L^2)}, \tag{1.29a}$$

$$\text{(v) } \|w - w_h^N\|_{H^1(0,T;L^2)} \le C(h + N^{-1/2})\|f\|_{H^{1/2}(0,T;L^2)}, \tag{1.29b}$$

and if $f \in H^{\alpha/2}(0,T;L^2)$ then

$$\text{(vi) } \|c - c_h^N\|_{L^2(0,T)} \le C(h + N^{-1/2})\|f\|_{H^{\alpha/2}(0,T;L^2)}. \tag{1.30}$$

Finally if Ω is a convex polygon, we have (i)′ $\|u - u_h^N\|_{L^2(0,T;L^2)} \le C(h^2 + N^{-1})\|f\|_{L^2(0,T;L^2)}$ *and the regular part w can be replaced by u in the above error estimates* (ii)–(v).

It is also noted that the error estimate (1.28a) shows a better L^2-error estimate for the regular part w in the space mesh-size h than that given in [7, Theorem 4.1].

This paper is organized as follows. In Section 2 we show the a priori estimates for the solution W of (1.21). In Section 3 we derive stability and the error estimate for the discrete problem (1.22) and show Theorem 1.6. In Section 4 we give numerical examples for the approximations: $\mathfrak{C}_h$, W_h, w_h^N and c_h^N, confirming the derived convergence rates.

2. The generalized boundary value problem (1.20)

In this section we show a unique existence of the solution for the problem (1.20), which is needed for showing the stability of the discrete problem (1.22) (see Theorem 3.3). We first give a useful lemma, which is used in giving an optimal dependency of the parameter λ for the a priori estimates for $\mathfrak{C}$ and W.

Lemma 2.1. *Let λ be a complex number with $\mathrm{Re}\,\lambda \ge 0$ and $\theta_\lambda = \mathrm{Arg}(\lambda)$ the argument of λ. Then the number $\gamma(\lambda)$ in (1.19) is bounded below: $|\gamma(\lambda)| \ge 1 - 8^{-1/2}$ and for $\forall a > -1$,*

$$\int_0^{\delta_1} e^{-\delta_2 \eta r} r^a dr \le C(1+|\lambda|)^{-(a+1)/2}, \tag{2.1}$$

where $\eta = |\lambda|^{1/2}\cos(\theta_\lambda/2)$ and δ_j are some positive numbers.

Proof. Clearly $\gamma(0) = 1$. We consider $\gamma(\lambda)$ for $\lambda \ne 0$. A direct calculation shows

$$\psi_\lambda^* = e^{-r\sqrt{\bar\lambda}} r^{-\alpha-1} \sin[\alpha(\theta-\omega_1)][(2\alpha-1)\sqrt{\bar\lambda}\chi_2 + (1-2\alpha-2r\sqrt{\bar\lambda})\chi_2' + r\chi_2''].\tag{2.2}$$

Recalling $\phi_\lambda = \chi_1 e^{-r\sqrt{\lambda}} r^\alpha \sin[\alpha(\theta-\omega_1)]$, we then have

$$\begin{aligned}\Big|\int_\Omega \phi_\lambda \overline{\psi_\lambda^*} d\mathbf{x}\Big| &\le (2\alpha-1)|\lambda|^{1/2}\int_0^{d_2} e^{-2\eta r}dr \int_{\omega_1}^{\omega_2} \sin^2[\alpha(\theta-\omega_1)]d\theta \\ &\le \frac{1}{4\eta}\,\omega(2\alpha-1)|\lambda|^{1/2} \le \pi/\sqrt{8},\end{aligned}$$

because $|e^{-r\sqrt{\lambda}}||e^{-r\sqrt{\bar\lambda}}| = e^{-2\eta r}$ and $\eta \ge |\lambda|^{1/2}/\sqrt{2}$ for $\theta_\lambda \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Hence we obtain that $|\gamma(\lambda)| \ge 1 - 1/\sqrt{8}$.

To show the second part. If $|\lambda| \geq 1$ then $|\lambda|^{-1} \leq 2/(1+|\lambda|)$, so one has

$$\begin{aligned}\int_0^{\delta_1} e^{-\delta_2\eta r} r^a dr &= (\delta_2\eta)^{-(a+1)} \int_0^{\delta_1\delta_2\eta} e^{-r_1} r_1^a dr_1 \quad (r_1 = \delta_2\eta r)\\ &\leq C|\lambda|^{-(a+1)/2}\\ &\leq C(1+|\lambda|)^{-(a+1)/2}.\end{aligned}$$

On the other hand, if $|\lambda| < 1$ then $1 < 2/(1+|\lambda|)$, so we have

$$\int_0^{\delta_1} e^{-\delta_2\eta r} r^a dr \leq \int_0^{\delta_1} r^a dr \leq C \leq C(1+|\lambda|)^{-(a+1)/2}.$$

Hence the required estimate follows. □

We next estimate the numbers $\mathfrak{C}_1(W)$ and $\mathfrak{C}_2(F)$ defined in (1.19).

Lemma 2.2. *Let* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda \geq 0$. *If* $W \in \mathrm{H}_0^1$ *and* $F \in \mathrm{L}^2$ *then*

$$|\mathfrak{C}_1(W)| \leq C(1+|\lambda|)^{\alpha/2}\|\nabla W\|_0, \tag{2.3}$$

$$|\mathfrak{C}_2(F)| \leq C(1+|\lambda|)^{(\alpha-1)/2}\|F\|_0. \tag{2.4}$$

Proof. Since $|\psi_\lambda^*| \leq C(1+|\lambda|)^{1/2} e^{-\eta r} r^{-\alpha-1}$ for $r \leq d_3$ and $\psi_\lambda^* = 0$ for $r > d_3$, one has

$$\begin{aligned}|\mathfrak{C}_1(W)| &\leq C(1+|\lambda|)^{1/2} \int_\Omega |r^{-1}W| e^{-\eta r} r^{-\alpha} d\mathbf{x}\\ &\leq C(1+|\lambda|)^{1/2}\|r^{-1}W\|_0 \|e^{-\eta r} r^{-\alpha}\|_0.\end{aligned}$$

By [20, Theorem 1.4.4.4], one has $\|r^{-1}W\|_0 \leq C\|\nabla W\|_0$ and by (2.1), we have

$$\|e^{-\eta r} r^{-\alpha}\|_0^2 \leq C \int_0^{d_3} e^{-2\eta r} r^{-2\alpha+1} dr \leq C(1+|\lambda|)^{\alpha-1}.$$

Combining above estimates, we get (2.3). Also the estimate (2.4) follows by

$$\|\psi_\lambda\|_0^2 \leq C \int_0^{d_3} e^{-2\eta r} r^{-2\alpha+1} dr \leq C(1+|\lambda|)^{\alpha-1}. \quad \square$$

For convenience we define the following norm depending on λ: for integer $j \geq 1$,

$$\|v\|_{j,\lambda} := \sqrt{\|v\|_j^2 + (1+|\lambda|)\|v\|_{j-1}^2}.$$

We next show Garding's type inequality for the bilinear form a_λ of (1.21).

Lemma 2.3. *There is a constant* C *independent of* λ *such that if* k *is a sufficiently large integer, then*

$$\left|a_\lambda(W,W) + k(1+|\lambda|)\|W\|_0^2\right| \geq C\|W\|_{1,\lambda}^2, \quad \forall W \in \mathrm{H}_0^1. \tag{2.5}$$

Also the bilinear form a_λ *is continuous on* $\mathrm{H}_0^1 \times \mathrm{H}_0^1$:

$$|a_\lambda(W,V)| \leq C\|W\|_{1,\lambda}\|V\|_{1,\lambda}, \quad \forall W, V \in \mathrm{H}_0^1.$$

Proof. For a large integer k one has

$$\begin{aligned}\left|a_\lambda(W,W) + k(1+|\lambda|)\|W\|_0^2\right| &\geq \|\nabla W\|_0^2 + (k + (k-1)|\lambda|)\|W\|_0^2 - |\mathfrak{C}_1(W)||\langle\phi_\lambda^*, W\rangle|\\ &\geq \|\nabla W\|_0^2 + (k-1)(1+|\lambda|)\|W\|_0^2 - C(1+|\lambda|)^{1/2}\|\nabla W\|_0\|W\|_0\\ &\geq 2^{-1}\|\nabla W\|_0^2 + (k-1-2^{-1}C^2)(1+|\lambda|)\|W\|_0^2,\end{aligned} \tag{2.6}$$

where the second inequality follows by (2.3) and

$$\|\phi_\lambda^*\|_0^2 \leq C(1+|\lambda|) \int_0^{d_2} e^{-2\eta r} r^{2\alpha-1} dr \leq C(1+|\lambda|)^{1-\alpha}, \tag{2.7}$$

and the third inequality follows by

$$C(1+|\lambda|)^{1/2}\|\nabla W\|_0\|W\|_0 \leq 2^{-1}\|\nabla W\|_0^2 + 2^{-1}C^2(1+|\lambda|)\|W\|_0^2.$$

Hence (2.5) follows by assuming that $k > 1 + 2^{-1}C^2$. The continuity of a_λ is clear. □

Using Lemma 2.3 we show a unique existence for (1.20) and the a priori estimate.

Theorem 2.4. *If $F \in \mathrm{L}^2$, there exists a unique solution $W \in \mathrm{H}^1_0$ of (1.21), satisfying the a priori estimate:*

$$(1+|\lambda|)^{1/2}\|\nabla W\|_0 + (1+|\lambda|)\|W\|_0 \le C\|F\|_0, \tag{2.8}$$

where C is a constant not depending on λ.

Proof. We first formulate the weak problem (1.21) by the relation: $A_\lambda W = F_\lambda$ in H^{-1} where A_λ is defined by

$$\langle A_\lambda W, V\rangle = a_\lambda(W, V), \quad \forall V \in \mathrm{H}^1_0.$$

Set $T_\lambda := A_\lambda + k(1+|\lambda|)\mathrm{I}$ for a sufficiently large k. By Lemma 2.3, T_λ is invertible on H^1_0. Since $T_\lambda W = F_\lambda + k(1+|\lambda|)W$ and letting $\mathrm{K} := k(1+|\lambda|)T_\lambda^{-1}$, we get

$$(\mathrm{I}-\mathrm{K})W = T_\lambda^{-1}F_\lambda. \tag{2.9}$$

Here $\mathrm{K} : \mathrm{L}^2 \mapsto \mathrm{L}^2$ is compact by (2.5). By the Fredholm alternative theorem (Ref. [27, Section 5.7]) the kernel of the operator $(\mathrm{I}-\mathrm{K})$ either is trivial or has finite nontrivial solutions. For a nontrivial solution W, $T_\lambda W = k(1+|\lambda|)W$. Then $A_\lambda W = 0$ in H^{-1}, that is, $a_\lambda(W, V) = 0,\ \forall V \in \mathrm{H}^1_0$. Taking $V = W + \mathfrak{C}_1\phi_\lambda \in \mathrm{H}^1_0$ and by the integration by parts,

$$\begin{aligned} 0 &= a_\lambda(W, W+\mathfrak{C}_1\phi_\lambda) \\ &= \langle \nabla W, \nabla(W+\mathfrak{C}_1\phi_\lambda)\rangle + \lambda\|W+\mathfrak{C}_1\phi_\lambda\|_0^2 - \mathfrak{C}_1\langle \Delta\phi_\lambda, W+\mathfrak{C}_1\phi_\lambda\rangle \\ &= \|\nabla(W+\mathfrak{C}_1\phi_\lambda)\|_0^2 + \lambda\|W+\mathfrak{C}_1\phi_\lambda\|_0^2. \end{aligned}$$

Since $\mathrm{Re}\,\lambda \ge 0$, $W + \mathfrak{C}_1\phi_\lambda \equiv 0$. Plugging $W = -\mathfrak{C}_1\phi_\lambda$ into the formula $\mathfrak{C}_1(W)$ in (1.19) and since $\int_\Omega \phi_\lambda \overline{\psi^*_\lambda} d\mathbf{x} = \pi(1-\gamma)$, we get

$$\mathfrak{C}_1\Big(1 + \frac{1}{\pi\gamma}\int_\Omega \phi_\lambda \overline{\psi^*_\lambda} d\mathbf{x}\Big) = \mathfrak{C}_1/\gamma = 0.$$

Hence $\mathfrak{C}_1 = 0$ and $W \equiv 0$, which is a contradiction. Hence (2.9) has a unique solution W, given by $W = (\mathrm{I}-\mathrm{K})^{-1}T_\lambda^{-1}F_\lambda$. Since $(\mathrm{I}-\mathrm{K})^{-1}$ is bounded on L^2 and $T_\lambda^{-1} : \mathrm{L}^2 \mapsto \mathrm{H}^1_0$ is bounded, we have

$$\begin{aligned} \|W\|_0 &\le \|(\mathrm{I}-\mathrm{K})^{-1}\|_{\mathrm{L}^2\mapsto\mathrm{L}^2}\|T_\lambda^{-1}F_\lambda\|_0 \\ &\le C(1+|\lambda|)^{-1}\|F_\lambda\|_0 \\ &\le C(1+|\lambda|)^{-1}\|F\|_0. \end{aligned}$$

Again, using $T_\lambda W = F_\lambda + k(1+|\lambda|)W$ and the inequality (2.5),

$$\begin{aligned} \|\nabla W\|_0^2 + (1+|\lambda|)\|W\|_0^2 &\le C|\langle T_\lambda W, W\rangle| \\ &\le C(|\langle F_\lambda, W\rangle| + k(1+|\lambda|)\|W\|_0^2) \\ &\le C(1+|\lambda|)^{-1}\|F\|_0^2. \end{aligned}$$

Hence (2.8) follows. □

We estimate the number $\mathfrak{C}$, which is the same as that given in Theorem 1.2.

Corollary 2.5. *The coefficient $\mathfrak{C}$ of (1.19) is estimated by*

$$(1+|\lambda|)^{(1-\alpha)/2}|\mathfrak{C}| \le C\|F\|_0 \tag{2.10}$$

for a constant C, not depending on λ.

Proof. Combining (2.3)–(2.4),

$$|\mathfrak{C}| \le C(1+|\lambda|)^{(\alpha-1)/2}\big[(1+|\lambda|)^{1/2}\|\nabla W\|_0 + \|F\|_0\big], \tag{2.11}$$

so the result follows by (2.8). □

3. The discrete problem (1.22) with parameter

In this section we show the stability of the discrete problem (1.22) and derive the error estimate. Uniqueness of the solution $W_h \in \mathcal{V}_h$ of (1.22) is obtained by Schatz's method concerning Ritz–Galerkin methods of indefinite bilinear forms (see [30]).

We next consider the adjoint problem of the bilinear form a_λ: $a^*_\lambda(z,w)=\overline{a_\lambda(w,z)}$ for all $w,z\in \mathrm{H}^1_0$. The adjoint problem is to find $z\in \mathrm{H}^1_0$ such that

$$a^*_\lambda(z,w)=\langle g,w\rangle,\quad \forall\, w\in \mathrm{H}^1_0, \tag{3.1}$$

where

$$a^*_\lambda(z,w)=\langle \nabla z,\nabla w\rangle+\overline{\lambda}\langle z,w\rangle-\mathfrak{C}^*_1(z)\langle \psi^*_\lambda,w\rangle,$$
$$\mathfrak{C}^*_1(z)=\frac{1}{\pi\gamma}\int_\Omega z\overline{\phi^*_\lambda}\,d\mathbf{x}.$$

It is recalled that problem (3.1) is the weak formulation for the following boundary value problem:

$$\begin{aligned}&-(\Delta-\overline{\lambda}\mathrm{I})z-\mathfrak{C}^*_1(z)\psi^*_\lambda=g\quad \text{in } \Omega,\\ &z=0\quad \text{on } \Gamma.\end{aligned} \tag{3.2}$$

Lemma 3.1. *There is a unique solution $z\in \mathrm{H}^1_0$ of (3.1) such that if $g\in \mathrm{H}^{s-2}$ for $s\in[1,2-\alpha)$, then*

$$\|z\|_s+(1+|\lambda|)^{1/2}\|z\|_{s-1}\le C\|g\|_{s-2}. \tag{3.3}$$

In particular, in the case $\lambda=0$ the solution z satisfies $\|z\|_s\le C\|g\|_{s-2}$ for $s\in[1,1+\alpha)$.

Proof. We first show the unique existence of z in H^1_0. Let $T^*_\lambda=A^*_\lambda+k(1+|\lambda|)\mathrm{I}$ for a large integer k where $\langle A^*_\lambda z,v\rangle=a^*_\lambda(z,v),\ \forall\, v\in \mathrm{H}^1_0$. As shown in (2.6), we have

$$\begin{aligned}|\langle T^*_\lambda z,z\rangle|&\ge \|\nabla z\|^2_0+(k-1)(1+|\lambda|)\|z\|^2_0-|\mathfrak{C}^*_1(z)\langle\psi^*_\lambda,z\rangle|\\ &\ge 2^{-1}\|\nabla z\|^2_0+(k-1-2^{-1}C^2)(1+|\lambda|)\|z\|^2_0,\end{aligned}$$

which is shown by

$$\begin{aligned}|\mathfrak{C}^*_1(z)\langle\psi^*_\lambda,z\rangle|&\le C(1+|\lambda|)^{1/2}\|\nabla z\|_0\|z\|_0\quad (\text{by (2.7) and (2.3)})\\ &\le 2^{-1}\|\nabla z\|^2_0+2^{-1}C^2(1+|\lambda|)\|z\|^2_0.\end{aligned}$$

For a large integer $k>1+2^{-1}C^2$, one has

$$|\langle T^*_\lambda z,z\rangle|\ge C[\|\nabla z\|^2_0+(1+|\lambda|)\|z\|^2_0],\quad \forall z\in \mathrm{H}^1_0. \tag{3.4}$$

By the relation $T^*_\lambda z=g+k(1+|\lambda|)z$, one has

$$(\mathrm{I}-\mathrm{K}^*)z=T^{*-1}_\lambda g, \tag{3.5}$$

where $\mathrm{K}^*:=k(1+|\lambda|)T^{*-1}_\lambda$. By (3.4), $\mathrm{K}^*:\mathrm{L}^2\mapsto \mathrm{L}^2$ is a compact operator. As shown in Theorem 2.4, the kernel of $\mathrm{I}-\mathrm{K}^*$ is trivial. If $(\mathrm{I}-\mathrm{K}^*)z=0$ for a nontrivial $z\in \mathrm{L}^2$ then $A^*_\lambda z=0$ in H^{-1}, that is,

$$a^*_\lambda(z,v)=a_\lambda(v,z)=0,\quad \forall\, v\in \mathrm{H}^1_0. \tag{3.6}$$

Taking $v=z-\mathfrak{C}_1\phi_\lambda\in \mathrm{H}^1_0$ in (3.6) and by integration by parts, we have

$$\begin{aligned}0&=a_\lambda(z-\mathfrak{C}_1\phi_\lambda,z)\\ &=\langle\nabla(z-\mathfrak{C}_1\phi_\lambda),\nabla z\rangle+\lambda\|z\|^2_0-\mathfrak{C}_1\langle\Delta\phi_\lambda,z\rangle\\ &=\|\nabla z\|^2_0+\lambda\|z\|^2_0.\end{aligned}$$

Since $\mathrm{Re}\,\lambda\ge 0$, we have $z\equiv 0$, which is a contradiction. Hence the formula $z=(\mathrm{I}-\mathrm{K}^*)^{-1}T^{*-1}_\lambda g$ by (3.5) shows unique existence in H^1_0. Since $(\mathrm{I}-\mathrm{K}^*)^{-1}$ is bounded on L^2 and $\|T^{*-1}_\lambda\|_{\mathrm{H}^{-1}\mapsto \mathrm{L}^2}\le C(1+|\lambda|)^{-1/2}$ by (3.4), we have

$$\begin{aligned}\|z\|_0&\le \|(\mathrm{I}-\mathrm{K}^*)^{-1}\|_{\mathrm{L}^2\mapsto \mathrm{L}^2}\|T^{*-1}_\lambda\|_{\mathrm{H}^{-1}\mapsto \mathrm{L}^2}\|g\|_{-1}\\ &\le C(1+|\lambda|)^{-1/2}\|g\|_{-1}.\end{aligned} \tag{3.7}$$

Using (3.4) and the relation: $T^*_\lambda z=g+k(1+|\lambda|)z$,

$$\begin{aligned}\|\nabla z\|^2_0+(1+|\lambda|)\|z\|^2_0&\le C|\langle T^*_\lambda z,z\rangle|\\ &\le C(|\langle g,z\rangle|+k(1+|\lambda|)\|z\|^2_0)\\ &\le C(\|z\|_1+(1+|\lambda|)^{1/2}\|z\|_0)\|g\|_{-1}.\end{aligned}$$

Hence the inequality (3.3) for $s=1$ follows.

To show the inequality (3.3) for $s > 1$. By the same procedure as used in showing (2.8), one has $(1+|\lambda|)^{1/2}\|z\|_1 \le C\|g\|_0$. By interpolation theory between integer values and using (3.7),

$$(1+|\lambda|)^{1/2}\|z\|_{s-1} \le C\|g\|_{s-2} \quad \text{for } s \in (1,2). \tag{3.8}$$

Note that an increased regularity for the solution z of problem (3.2) mainly depends on the following regularity of the function ψ_λ^* in (2.2):

$$\|\psi_\lambda^*\|_{s-2} \le C(1+|\lambda|)^{(s+\alpha-1)/2} \quad \text{for } s < 2-\alpha \ (\lambda \ne 0),$$
$$\|\psi_\lambda^*\|_{s-2} \le C \quad \text{for } s < 1+\alpha \ (\lambda = 0).$$

By Remark 3.2 and for $1 < s < \min\{1+\alpha, 2-\alpha\}$, one has

$$\begin{aligned}\|z\|_s + (1+|\lambda|)^{1/2}\|z\|_{s-1} &\le C(|\mathfrak{C}_1^*(z)|\|\psi_\lambda^*\|_{s-2} + \|g\|_{s-2})\\ &\le C[(1+|\lambda|)^{1/2}\|z\|_{s-1} + \|g\|_{s-2}]\\ &\le C\|g\|_{s-2} \quad (\text{by } (3.8)),\end{aligned} \tag{3.9}$$

because $|\mathfrak{C}_1^*(z)| \le C\|z\|_{s-1}\|\phi_\lambda^*\|_{1-s} \le C(1+|\lambda|)^{(2-s-\alpha)/2}\|z\|_{s-1}$, which follows by $|\phi_\lambda^*| \le C(1+|\lambda|)^{1/2}e^{-\eta r}r^{\alpha-1}$ where η is defined in Lemma 2.1. □

Remark 3.2. Let $z \in \mathrm{H}_0^1$ be the solution for the problem: $-\Delta z + \lambda z = g \in \mathrm{H}^{s-2}$. Let $u = \chi z$ for the cutoff function χ. Then u satisfies the regularity

$$\|u\|_s + (1+|\lambda|)^{1/2}\|u\|_{s-1} \le C\|g\|_{s-2} \quad \text{for } s \in [1, 1+\alpha).$$

Indeed, as shown in (3.8) we have $(1+|\lambda|)^{1/2}\|u\|_{s-1} \le C\|g\|_{s-2}$ for $s \in [1,2]$. We now show $\|u\|_s \le C\|g\|_{s-2}$ for $s \in [1, 1+\alpha)$. We write $u(r) = u(r,\cdot)$. By [31, Theorem 2.1] we use the Mellin transform

$$\breve{u}(\zeta) = \frac{1}{\sqrt{2\pi}}\int_0^\infty r^{i\zeta-1}u(r)dr, \quad \zeta = \sigma + i\eta \in \mathbb{C}.$$

Since $u \in \mathrm{H}_0^1$, $\breve{u}$ is analytic for $\eta = \operatorname{Im}\zeta < 0$ and square integrable for $\eta \le 0$. Let $g_1 = g - \lambda u$. By the integration by parts, one has $(\zeta^2 - \partial_{\theta\theta})\breve{u}(\zeta) = \breve{g}_1(\zeta - 2i)$. Since $(\zeta^2 - \partial_{\theta\theta})^{-1}$ has no pole for $\eta = 0$, and by the inverse Mellin transform, the solution u is expressed by

$$u(r) = \frac{1}{\sqrt{2\pi}}\int_{-\infty+i0}^{\infty+i0} r^{-i\zeta}(\zeta^2 - \partial_{\theta\theta})^{-1}\breve{g}_1(\zeta - 2i)d\zeta.$$

For $s \in [1, 1+2\alpha)$ and $s \ne 1+\alpha$, we define u_{R} by

$$u_{\mathrm{R}}(r) = \frac{1}{\sqrt{2\pi}}\int_{-\infty+i(s-1)}^{\infty+i(s-1)} r^{-i\zeta}(\zeta^2 - \partial_{\theta\theta})^{-1}\breve{g}_1(\zeta - 2i)d\zeta.$$

If we assume that $g_1 \in \mathrm{H}^{s-2}$, then $g_1(\zeta - 2i)$ is square integrable for $\eta \le s-1$, and by the Residue theorem,

$$u - u_{\mathrm{R}} = \begin{cases} 0 & \text{for } s-1 < \alpha,\\ u_{\mathrm{S}} & \text{for } s-1 > \alpha,\end{cases} \tag{3.10}$$

where, by [23, Section II.2.2.1],

$$\begin{aligned} u_{\mathrm{S}} &= \sqrt{2\pi}i \operatorname*{Res}_{\zeta = i\alpha} r^{-i\zeta}(\zeta^2 - \partial_{\theta\theta})^{-1}\breve{g}_1(\zeta - 2i)\\ &= r^\alpha \lim_{\zeta \to i\alpha}(\zeta^2 - \partial_{\theta\theta})^{-1}[(\zeta - i\alpha)\breve{g}_1(\zeta - 2i)]\\ &= c_1 r^\alpha \sin[\alpha(\theta - \omega_1)]\end{aligned}$$

for a constant $c_1 \ne 0$. The remainder satisfies $\|u_{\mathrm{R}}\|_2 \le C\|g_1\|_0 \le C\|g\|_0$ by $(1+|\lambda|)\|u\|_0 \le C\|g\|_0$. Since $u = u_{\mathrm{R}}$ for $1 \le s < 1+\alpha$ by (3.10), we have $\|u\|_s = \|u_{\mathrm{R}}\|_s \le C\|g\|_{s-2}$ for $s \in (1, 1+\alpha)$ by the interpolation theory between integer values. Hence the required estimate follows. □

It is assumed that there is a projection operator $\Pi_h : \mathrm{H}^1 \mapsto \mathcal{V}_h$ (see [32]) satisfying the following interpolation error estimates:

$$\begin{aligned}\|v - \Pi_h v\|_0 &\le Ch^s\|v\|_s \quad \text{for } v \in \mathrm{H}^s \ (0 \le s \le 2),\\ \|v - \Pi_h v\|_1 &\le Ch^{s-1}\|v\|_s \quad \text{for } v \in \mathrm{H}^s \ (1 \le s \le 2),\end{aligned} \tag{3.11}$$

where C is a constant independent of h. For the examples of (3.11), one may refer to Refs. [32–34].

We show the a priori estimate for $W_h \in \mathcal{V}_h$ and a number $\mathfrak{C}_h$.

Theorem 3.3. *Let $s \in (1, 2-\alpha)$. Suppose the mesh-size h is sufficiently small: $h < [Ck(1+|\lambda|)]^{-\frac{1}{2(s-1)}}$ for a constant C, where k is the number in Lemma 2.3. If $F \in L^2$, there exists a unique solution $W_h \in \mathcal{V}_h$ of (1.22), satisfying*

$$\|W_h\|_{1,\lambda} \le C\|W\|_{1,\lambda} \le C(1+|\lambda|)^{-1/2}\|F\|_0, \tag{3.12}$$

where C is a constant not depending on h. Also the approximation $\mathfrak{C}_h$ in (1.23) is uniquely defined:

$$(1+|\lambda|)^{(1-\alpha)/2}|\mathfrak{C}_h| \le C(1+|\lambda|)^{1/2}\|W\|_{1,\lambda} + C\|F\|_0 \le C\|F\|_0. \tag{3.13}$$

Proof. Let $e = W - W_h$ for the solution W of (1.21) and the discrete solution W_h of (1.22). We claim that for $s \in (1, 2-\alpha)$,

$$\|e\|_0 \le Ch^{s-1}\|e\|_{1,\lambda}. \tag{3.14}$$

To show this, consider the dual problem: $a^*_\lambda(z, v) = \langle e, v\rangle, \ \forall v \in \mathcal{V}$. By Lemma 3.1 this solution z satisfies, for $s \in (1, 2-\alpha)$,

$$\|z\|_s + (1+|\lambda|)^{1/2}\|z\|_{s-1} \le C\|e\|_{s-2} \le C\|e\|_0.$$

Taking $v = e$ in the dual problem and using the orthogonality $a_\lambda(e, v_h) = 0, \ \forall v_h \in \mathcal{V}_h$,

$$\begin{aligned}\|e\|_0^2 &= |a^*_\lambda(z, e)| = |a_\lambda(e, z - \Pi_h z)| \\ &\le C\|e\|_{1,\lambda}\|z - \Pi_h z\|_{1,\lambda} \\ &\le Ch^{s-1}(\|z\|_s + (1+|\lambda|)^{1/2}\|z\|_{s-1})\|e\|_{1,\lambda} \\ &\le Ch^{s-1}\|e\|_0\|e\|_{1,\lambda}.\end{aligned}$$

Hence (3.14) follows. We next show (3.12). Using (2.5) and (3.14), we have

$$\begin{aligned}\|e\|_{1,\lambda}^2 &\le C\left|a_\lambda(e, e) + k(1+|\lambda|)\|e\|_0^2\right| \\ &= C\left|a_\lambda(e, W - \Pi_h W + \Pi_h W - W_h) + k(1+|\lambda|)\|e\|_0^2\right| \\ &\le C\|e\|_{1,\lambda}\|W - \Pi_h W\|_{1,\lambda} + Ck(1+|\lambda|)h^{2(s-1)}\|e\|_{1,\lambda}^2.\end{aligned}$$

If the mesh-size h is sufficiently small then

$$\|e\|_{1,\lambda} \le C\|W - \Pi_h W\|_{1,\lambda} \le C\|W\|_{1,\lambda}. \tag{3.15}$$

Using (3.15) we have $\|W_h\|_{1,\lambda} \le \|W\|_{1,\lambda} + \|e\|_{1,\lambda} \le C\|W\|_{1,\lambda}$. The second inequality of (3.12) follows by (2.8). Finally (3.13) can be shown by the procedure as done in (2.11) and the estimate (3.12). □

Next we derive the error estimates of W_h and $\mathfrak{C}_h$.

Theorem 3.4. *Let $W \in \mathcal{V}$ be the solution of (1.21) and $W_h \in \mathcal{V}_h$ the solution of (1.22). With the conditions of Theorem 3.3 there is a constant C not depending on h such that, for $0 < \epsilon \ll 1-\alpha$,*

$$\|W - W_h\|_0 \le Ch^{2-\alpha-\epsilon}\|F\|_0, \tag{3.16a}$$

$$\|\nabla(W - W_h)\|_0 + (1+|\lambda|)^{1/2}\|W - W_h\|_0 \le Ch\|F\|_0 \tag{3.16b}$$

and also the number $\mathfrak{C}_h$ in (1.23) satisfies

$$|\mathfrak{C} - \mathfrak{C}_h| \le C(1+|\lambda|)^{\frac{\alpha}{2}}h\|F\|_0. \tag{3.17}$$

In particular it is noted that if the parameter λ is set to be zero in the problems (1.21) and (1.22), one has the same result as [14, Theorem 4.4]: $\|W - W_h\|_0 \le Ch^{1+\alpha-\epsilon}\|F\|_0$ and $|\mathfrak{C} - \mathfrak{C}_h| \le Ch^{1+\alpha-\epsilon}\|F\|_0$ for any ϵ with $0 < \epsilon \ll \alpha$.

Proof. By (3.15) and (3.11), the error $e = W - W_h$ satisfies

$$\begin{aligned}\|e\|_{1,\lambda} &\le Ch(\|W\|_2 + (1+|\lambda|)^{1/2}\|W\|_1) \\ &\le Ch\|F\|_0,\end{aligned}$$

which shows (3.16b). Let $0 < \epsilon \ll 1-\alpha$ be any number. Using (3.14) with $s = 2-\alpha-\epsilon$ and (3.16b), $\|e\|_0 \le Ch^{1-\alpha-\epsilon}\|e\|_{1,\lambda} \le Ch^{2-\alpha-\epsilon}\|F\|_0$. By (2.3) and (3.16b), we have

$$\begin{aligned}|\mathfrak{C} - \mathfrak{C}_h| &= |\mathfrak{C}_1(e)| \le C(1+|\lambda|)^{\alpha/2}\|\nabla e\|_0 \\ &\le C(1+|\lambda|)^{\alpha/2}h\|F\|_0.\end{aligned}$$

If $\lambda = 0$, the inequality (3.9) is true for all $s \in [1, 1+\alpha)$ and (3.14) holds for $s \in [1, 1+\alpha)$. So the required result for $\|e\|_0$ follows by (3.16b). Also, one has $|\mathfrak{C} - \mathfrak{C}_h| \le C\|e\|_0 \le Ch^{1+\alpha-\epsilon}\|F\|_0$ since $\psi^*_\lambda \in L^2$ for $\lambda = 0$. □

Finally we show the main result: Theorem 1.6.

Proof of Theorem 1.6. Set $\beta_{t,k} = (1+|\lambda_k|)^t$. We first show the a priori estimate for w_h^N and c_h^N of (1.27). By (3.12), $\beta_{1/2,k}\|W_{k,h}\|_1 + \beta_{1,k}\|W_{k,h}\|_0 \le C\|F_k\|_0$ for $|k| \le N$. Then

$$\|w_h^N\|^2_{H^{1/2}(0,T;H^1)} \le CT \sum_{|k|\le N} \|F_k\|_0^2 \le CT \sum_{k=-\infty}^{\infty} \|F_k\|_0^2 = C\|f\|^2_{L^2(0,T;L^2)}. \tag{3.18}$$

As shown in (3.18) we have $\|c_h^N\|_{H^{(1-\alpha)/2}(0,T)} \le C\|f\|_{L^2(0,T;L^2)}$ since $\beta_{(1-\alpha)/2,k}|\mathfrak{C}_{k,h}| \le C\|F_k\|_0,\ \forall\,|k| \le N$ obtained by (3.13). Clearly $\|w_h^N\|_{H^1(0,T;L^2)} \le C\|f\|_{L^2(0,T;L^2)}$. Hence the a priori estimate follows.

To derive the error estimates. By (2.8) one sees that for $|k| > N$,

$$\|W_k\|_0 \le C\beta_{-1,k}\|F_k\|_0, \tag{3.19a}$$

$$\|W_k\|_1 \le C\beta_{-1/2,k}\|F_k\|_0. \tag{3.19b}$$

Using (3.19a) and the estimate (3.16a): $\|W_k - W_{k,h}\|_0 \le Ch^{2-\alpha-\epsilon}\|F_k\|_0$, we have

$$\begin{aligned}\|w - w_h^N\|^2_{L^2(0,T;L^2)} &= T\sum_{|k|\le N}\|W_k - W_{k,h}\|_0^2 + T\sum_{|k|>N}\|W_k\|_0^2 \\ &\le Ch^{2(2-\alpha-\epsilon)}T\sum_{|k|\le N}\|F_k\|_0^2 + CT\sum_{|k|>N}\beta_{-2,k}\|F_k\|_0^2 \\ &\le C(h^{2(2-\alpha-\epsilon)} + N^{-2})\|f\|^2_{L^2(0,T;L^2)}.\end{aligned} \tag{3.20}$$

Then (1.28a) follows. As done in (3.20), by (3.19b) and the estimate (3.16b): $\|W_k - W_{k,h}\|_1 \le Ch\|F_k\|_0$, (1.28b) is shown. Also, since $\beta_{1/2,k}\|W_k - W_{k,h}\|_0 \le Ch\|F_k\|_0$ (cf. (3.16b)) and $\beta_{1/2,k}\|W_k\|_0 \le CN^{-1/2}\|F_k\|_0$ (cf. (3.19a)), the inequality (1.28c) is derived. Similarly the error estimates (1.29a) and (1.29b) follow.

Since $|\mathfrak{C}_k - \mathfrak{C}_{k,h}| \le Ch\beta_{\alpha/2,k}\|F_k\|_0$ (cf. (3.17)) and $|\mathfrak{C}_k| \le C\beta_{(\alpha-1)/2,k}\|F_k\|_0 \le CN^{-1/2}\beta_{\alpha/2,k}\|F_k\|_0$ (cf. (2.10)), (1.30) follows. □

4. Numerical simulations

In this section we confirm the convergence rates given in Theorems 1.4 and 1.6 by suitable examples and list the numerical values of the convergence rates for the stress intensity factor and the regular part. The finite element code used in [35] has been modified for this numerical simulations.

Let $\mathcal{T}_h = \{T : T \text{ is a triangle in } \Omega\}$ be a regular triangulation of Ω with the mesh-size $h = \max_{T\in\mathcal{T}_h}\operatorname{diam}(T)$. Let $\mathcal{V}_h = \{v_h \in C(\Omega) : v_h|_T \in \mathbb{P}_1(T),\ \forall\,T \in \mathcal{T}_h,\ v_h|_\Gamma = 0\}$ be the finite dimensional space where $\mathbb{P}_1(T)$ denotes the space of linear functions on the triangle T.

4.1. The Poisson problem (1.12) with parameter

We compute the approximations W_h and $\mathfrak{C}_h$ by (1.22) and (1.23) on the L-shaped domain Ω depicted in Fig. 2(a). The singular functions for (1.15) are given by

$$\phi_\lambda = e^{-r\sqrt{\lambda}}\chi_1(r)r^{\frac{2}{3}}\sin\theta_1, \qquad \psi_\lambda = e^{-r\sqrt{\bar\lambda}}\chi_2(r)r^{-\frac{2}{3}}\sin\theta_1, \tag{4.1}$$

where $\theta_1 := \frac{2}{3}(\theta + \frac{\pi}{2})$ and

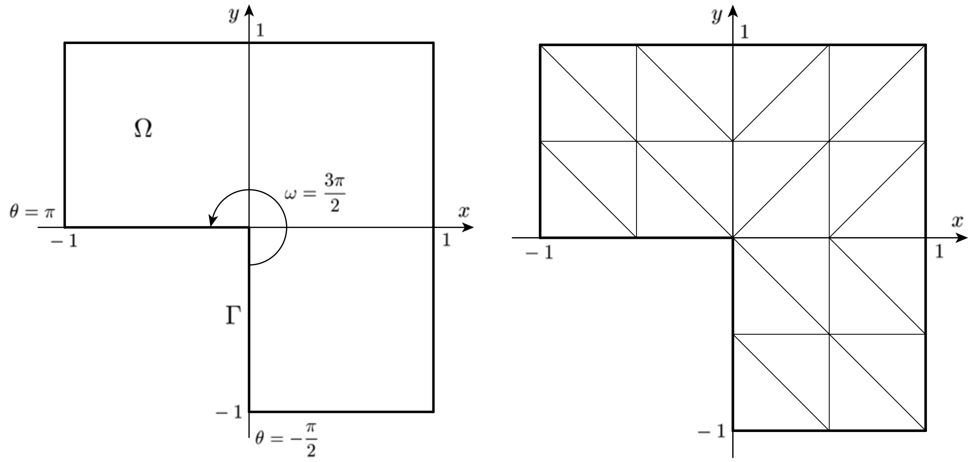
$$\chi_j = \begin{cases} 1 & \text{for } r \le j/4, \\ \dfrac{15}{16}\Big(\dfrac{8}{15} - \xi_j + \dfrac{2}{3}\xi_j^3 - \dfrac{1}{5}\xi_j^5\Big) & \text{for } j/4 < r < j/2, \\ 0 & \text{for } r \ge j/2, \end{cases}$$

with $\xi_j(r) = 8r/j - 3$ for $j = 1, 2$. Clearly χ_1 and χ_2 are in $C^2(\mathbb{R})$. Also let $\phi_\lambda^* = (\Delta - \lambda I)\phi_\lambda$.

We use the Sherman–Morrison (SM) formula in [15] to solve the matrix problem generated by the discrete problem (1.22). The procedure is as follows: Let $\{\varphi_j,\ j = 1,\dots,M\}$ be the nodal basis of $\mathcal{V}_h$. If $W_h = \sum_{j=1}^M w_j\varphi_j$, the matrix problem for (1.22) becomes

$$(\mathbf{A}_\lambda - \mathbf{B}_\lambda)\mathbf{w} = \mathbf{F}_\lambda, \tag{4.2}$$

where $\mathbf{A}_\lambda$ and $\mathbf{B}_\lambda$ are the $M\times M$-matrices with (i,j)-components: $\langle\nabla\varphi_j, \nabla\varphi_i\rangle + \lambda\langle\varphi_j, \varphi_i\rangle$ and $\mathfrak{C}_1(\varphi_j)\langle\phi_\lambda^*, \varphi_i\rangle$, respectively, $\mathbf{w} = \{w_j\}_{j=1}^M$ the unknown vector and $\mathbf{F}_\lambda = \{\langle F_\lambda, \varphi_j\rangle\}_{j=1}^M$ the load vector. Note that $\mathbf{A}_\lambda$ is invertible for $\operatorname{Re}\lambda \ge 0$ and $\mathbf{B}_\lambda = \mathbf{pq}^t$ is a



(a) L-shaped domain Ω.　　(b) Initial triangulation of Ω.

Fig. 2. The L-shaped domain Ω.

Table 1
Numerical errors for $\lambda = 1$.

h_j	nvar	$\lvert\mathfrak{C} - \mathfrak{C}_h\rvert$	Rate	$\lVert W - W_h\rVert_0$	Rate	$\lvert W - W_h\rvert_1$	Rate
2^{-1}	5	2.7732E−2		4.7274E−2		3.2654E−1	
2^{-2}	33	6.8882E−3	2.01	1.6808E−2	1.49	1.9790E−1	0.72
2^{-3}	161	1.7813E−3	1.95	4.6458E−3	1.86	1.0476E−1	0.92
2^{-4}	705	4.2972E−4	2.05	1.1925E−3	1.96	5.3178E−2	0.98
2^{-5}	2 945	1.1098E−4	1.95	3.0016E−4	1.99	2.6692E−2	0.99
2^{-6}	12,033	2.6277E−5	2.08	7.5147E−5	2.00	1.3359E−2	1.00
2^{-7}	48,641	7.0255E−6	1.90	1.8800E−5	2.00	6.6810E−3	1.00
2^{-8}	195,585	1.7137E−6	2.04	4.7000E−6	2.00	3.3407E−3	1.00
2^{-9}	784,385	4.2502E−7	2.01	1.1750E−6	2.00	1.6704E−3	1.00
2^{-10}	3,141,633	1.0634E−7	2.00	2.9375E−7	2.00	8.3519E−4	1.00

nonsymmetric and rank-one matrix with $\mathbf{p} = \{\langle \phi_\lambda^*, \varphi_j\rangle\}_{j=1}^M$ and $\mathbf{q} = \{\mathfrak{C}_1(\varphi_j)\}_{j=1}^M$. By the SM formula the inverse of $\mathbf{A}_\lambda - \mathbf{B}_\lambda$ is

$$(\mathbf{A}_\lambda - \mathbf{B}_\lambda)^{-1} = \mathbf{A}_\lambda^{-1} + \frac{\mathbf{A}_\lambda^{-1}\mathbf{p}\mathbf{q}^t\mathbf{A}_\lambda^{-1}}{1 - \mathbf{q}^t\mathbf{A}_\lambda^{-1}\mathbf{p}}.$$

Hence the solution $\mathbf{w} = (\mathbf{A}_\lambda - \mathbf{B}_\lambda)^{-1}\mathbf{F}_\lambda$ of system (4.2) is computed by the following procedure, called Algorithm $\mathcal{A}$:

1. Find $\mathbf{x}$ and $\mathbf{y}$ solving $\mathbf{A}_\lambda\mathbf{x} = \mathbf{F}_\lambda$ and $\mathbf{A}_\lambda\mathbf{y} = \mathbf{p}$, respectively.
2. Compute $a = 1/(1 - \mathbf{q}^t\mathbf{y})$ and $b = \mathbf{q}^t\mathbf{x}$.
3. Set $\mathbf{w} = \mathbf{x} + ab\,\mathbf{y}$.

Using this procedure we test three numerical examples:

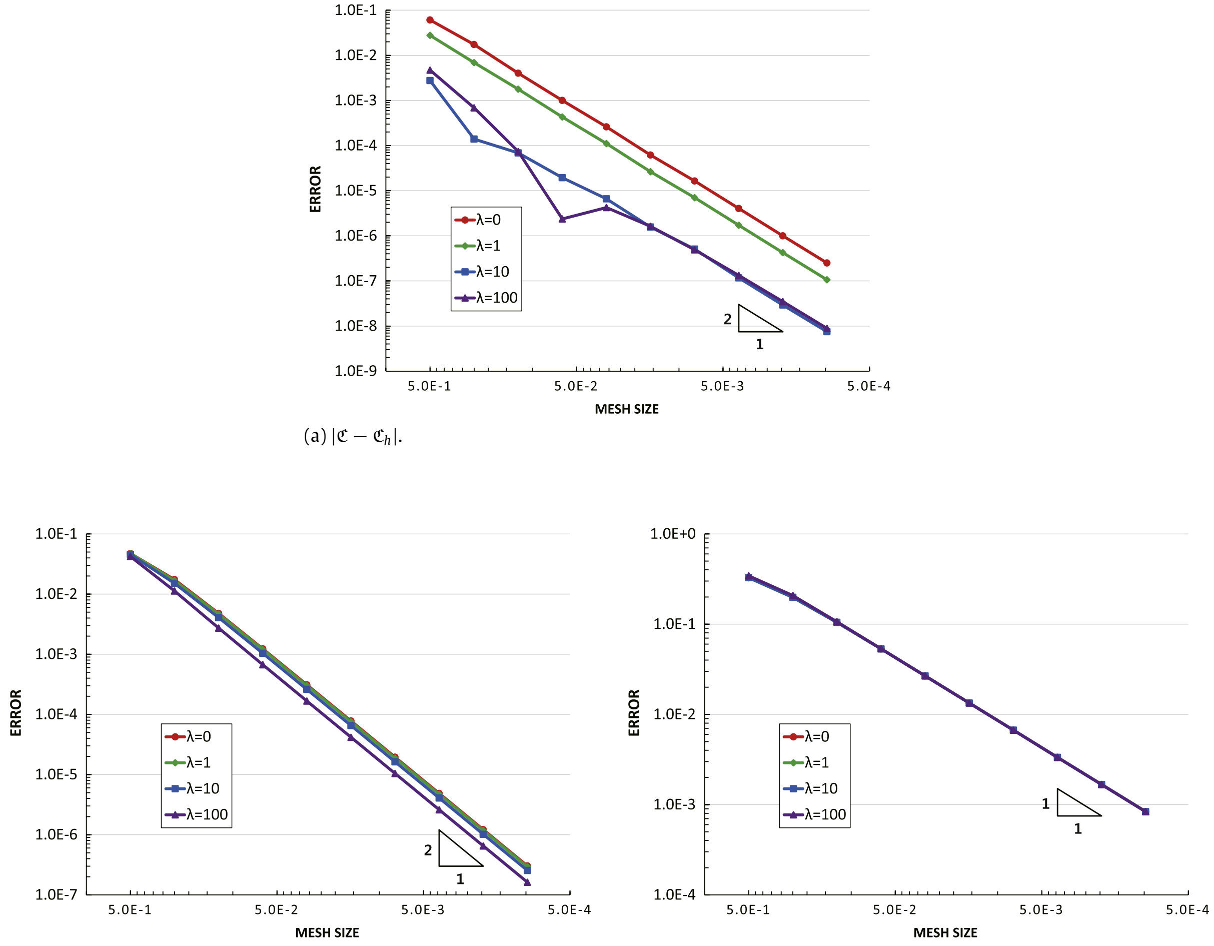
Example 1. As an exact solution of (1.12) we choose

$$\mathfrak{C} = 1, \qquad W(x, y) = (x - x^3)(y^2 - y^4). \tag{4.3}$$

With ϕ_λ in (4.1) we have $U = \mathfrak{C}\phi_\lambda + W$ and $F = -\Delta U + \lambda U$ is determined. With $F_\lambda = \mathfrak{C}_2(F)\phi_\lambda^* + F$ we solve the discrete problem (1.22) for W_h on the regular triangulation $\mathcal{T}_h$. Let $h_j = 2^{-j}$ be the mesh-size for each level $j \geq 1$. Let *nvar* be the number of unknowns and *Rate* the convergence rate. The rate is defined by $Rate := \log_2(e_{j-1}/e_j)$, where e_j denotes the corresponding error on the jth level.

In Table 1 we list the numerical values for errors and their convergence rates when $\lambda = 1$. Let $|v|_1 = \|\nabla v\|_0$. As predicted in (1.25) the numerical numbers for $|W - W_h|_1$ show the predicted convergence rate 1 but those for $\|W - W_h\|_0$ show much greater convergence rate 2 than the value $2 - \alpha - \epsilon \approx 4/3$ predicted in (1.25a). This is presumed from a sufficient smoothness, that is, H^2-regularity, of the exact solution W in (4.3) (Ref. [13, Section 6]). Also, if $\|W - W_h\|_0$ shows such a super-convergence, the error $|\mathfrak{C} - \mathfrak{C}_h|$ shows the same convergence rate 2 by the formula $\mathfrak{C}$.

In Fig. 3 we display the numerical errors for the parameters: $\lambda = 0, 1, 10$ and 100. The x-axis reads the value of mesh-size and the y-axis the value of corresponding errors. The computed error becomes smaller for the larger value λ and also shows the predicted convergence rates.



(a) $|\mathfrak{C}-\mathfrak{C}_h|$.

(b) $\|W-W_h\|_0$.

(c) $|W-W_h|_1$.

Fig. 3. Graphs of errors for $\lambda = 0, 1, 10$ and 100.

Example 2. We here simply test the Poisson problem (1.12) with parameter when $F = 1$. Then we do not have the exact formulae for the functions $U, \mathfrak{C}$ and W in the decomposition $U = \mathfrak{C}\phi_\lambda + W$. We consider the following numerical errors for level $j \geq 2$:

$$\mathcal{E}_{\mathfrak{C}} = |\mathfrak{C}_h^j - \mathfrak{C}_h^{j-1}|, \qquad \mathcal{E}_{W,0} = \|W_h^j - W_h^{j-1}\|_0,$$
$$\mathcal{E}_{W,1} = |W_h^j - W_h^{j-1}|_1, \tag{4.4}$$

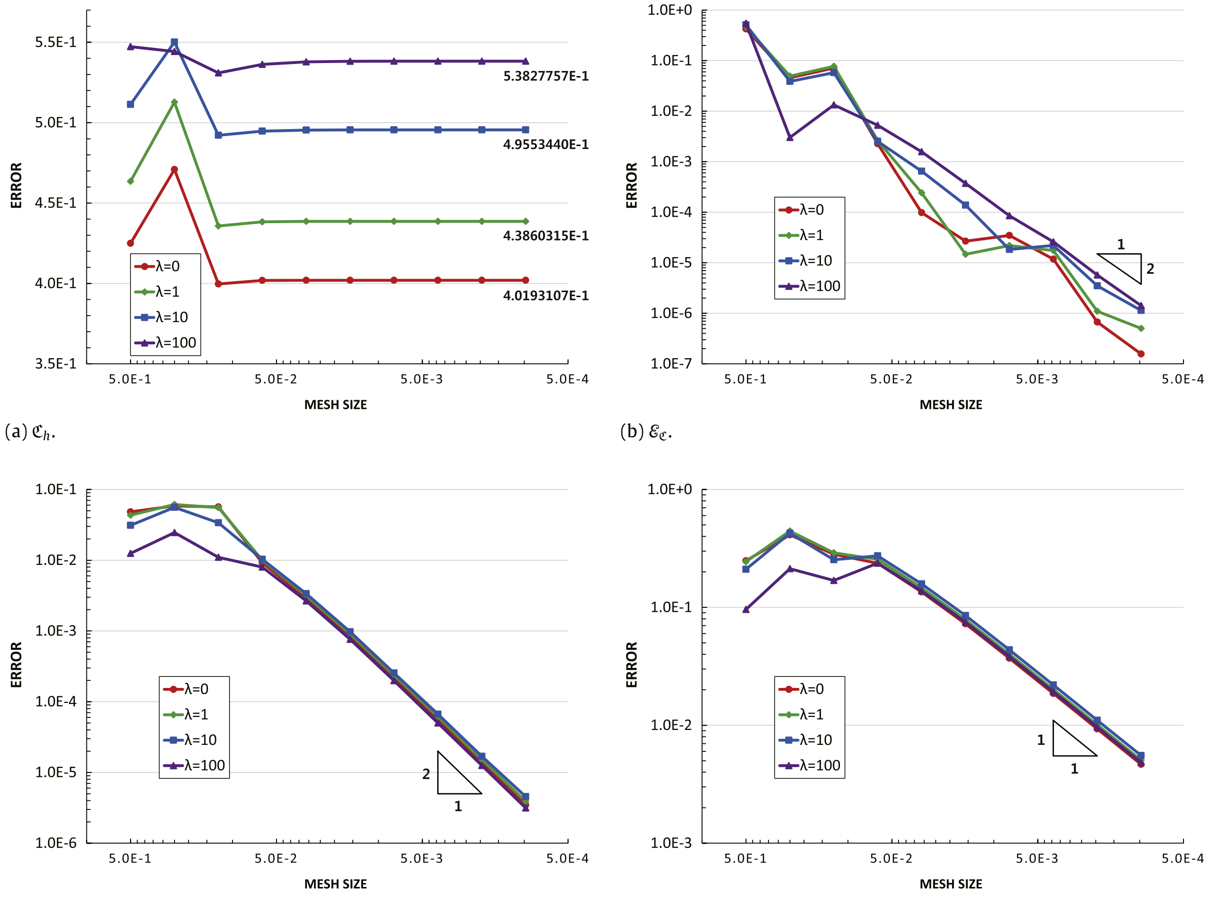
where $\mathfrak{C}_h^j$ and W_h^j denote the approximations on the mesh-size h_j.

In Fig. 4 we plot the approximation $\mathfrak{C}_h$ and the errors in (4.4) for $\lambda = 0, 1, 10$ and 100. As shown in Fig. 4(c)–(d) the errors $\mathcal{E}_{W,0}$ and $\mathcal{E}_{W,1}$ show the predicted convergence rates 2 and 1, explained in Example 1. In Fig. 4(b) we show the convergence rates of the error $\mathcal{E}_{\mathfrak{C}}$ for the different parameters. The rate for $\lambda = 1$ is irregular but much greater than the predicted rate 1 in (1.25c). This irregularity is due to the singular term

$$\psi_\lambda^* = (\Delta - \overline{\lambda}\mathrm{I})(e^{-r\sqrt{\overline{\lambda}}}\psi) = e^{-r\sqrt{\overline{\lambda}}}\Delta\psi + 2\nabla(e^{-r\sqrt{\overline{\lambda}}})\cdot\nabla\psi + [(\Delta - \overline{\lambda}\mathrm{I})(e^{-r\sqrt{\overline{\lambda}}})]\psi$$
$$\sim r^{-\alpha-1} \quad (\text{since } \Delta\psi \sim 0 \text{ near the origin}),$$

which appears in $\mathfrak{C}_1(W)$ of (1.19), i.e.,

$$\mathfrak{C}_1(W) \sim \int_\Omega W\overline{\psi_\lambda^*}d\mathbf{x} \sim \int_0^1 Wr^{-\alpha}dr.$$



(a) $\mathfrak{C}_h$.

(b) $\mathcal{E}_{\mathfrak{C}}$.

(c) $\mathcal{E}_{W,0}$.

(d) $\mathcal{E}_{W,1}$.

Fig. 4. Graphs of $\mathfrak{C}_h$ and errors for Example 2.

Example 3. Here we test the program with the complex parameters $\lambda = 2\pi ki,\ k \in \mathbb{Z}$, which are used in the next subsection for the Heat equation. The considered exact functions for $U = \mathfrak{C}\phi_\lambda + W$ are

$$\mathfrak{C} = e^{-\lambda}\lambda^{-3}[\lambda + 2 + e^{\lambda}(\lambda - 2)], \qquad W(x, y) = \mathfrak{C}(x - x^3)(y^2 - y^4).$$

In Table 2 we list the errors and their convergence rates when $\lambda = 2\pi i$. As predicted we have the rate $O(h)$ for the error $|W - W_h|_1$ and the super-convergence rate $O(h^2)$ for $|\mathfrak{C} - \mathfrak{C}_h|$ and $\|W - W_h\|_0$, which is assumed to be the same reason as explained in Example 1. In Fig. 5 we show and compare the errors for the different parameters: $\lambda = 2\pi ki$, $k = 0, 1, 10$ and 100.

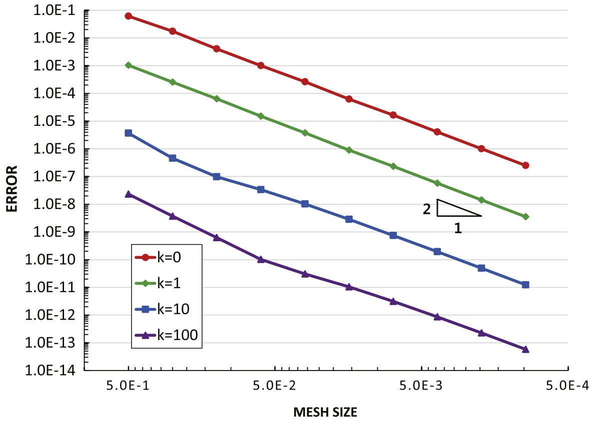
4.2. The Heat equation on the L-shaped domain

In this subsection we test the Heat equation on the L-shaped domain by using the FFEM. Take $\mathrm{T} = 1$ and set $\lambda_k = 2\pi ki$. The exact functions in the decomposition: $u = (\mathcal{E} \star c)(t)\phi + w$ are chosen by

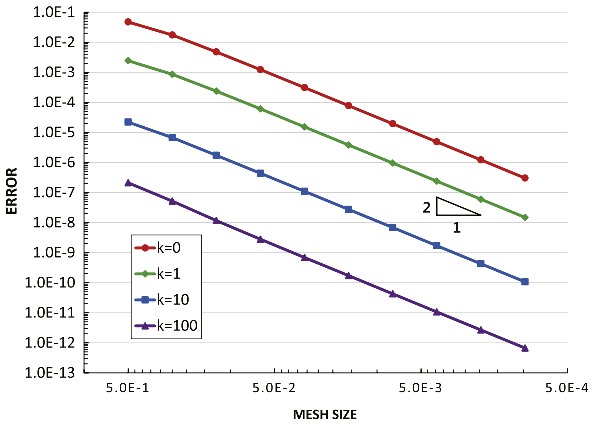
$$w(x, y, t) = c(t)(x - x^3)(y^2 - y^4), \quad c(t) = t(1 - t) = \sum_{k=-\infty}^{\infty} \mathfrak{C}_k e^{\lambda_k t},$$

$$\mathfrak{C}_k = \int_0^1 c(t)e^{-\lambda_k t}dt, \qquad (\mathcal{E} \star c)\phi = \sum_{k=-\infty}^{\infty} \mathfrak{C}_k e^{\lambda_k t}\phi_{\lambda_k},$$

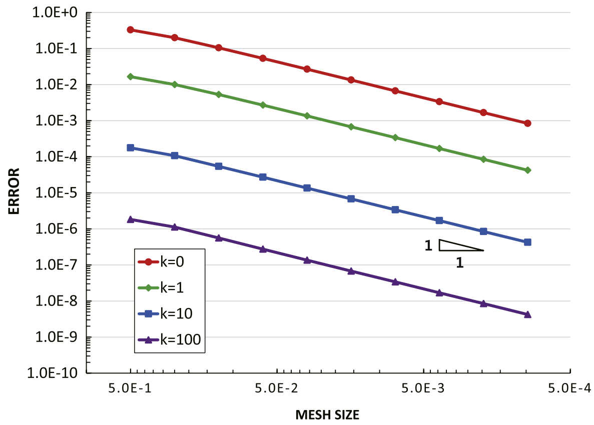
where ϕ_{λ_k} is given by (1.15) for $\lambda = \lambda_k$. Also we have $W_k(x, y) = \mathfrak{C}_k(x - x^3)(y^2 - y^4)$ and $F_k = -(\Delta - \lambda_k \mathrm{I})[\mathfrak{C}_k\phi_{\lambda_k} + W_k]$. For $\lambda = \lambda_k$ in Algorithm $\mathcal{A}$, let $\mathbf{A}_{\lambda_k}$ and $\mathbf{B}_{\lambda_k}$ be the $M \times M$-matrices with (i, j)-components: $\langle \nabla\varphi_j, \nabla\varphi_i\rangle + \lambda_k\langle\varphi_j, \varphi_i\rangle$ and $\mathfrak{C}_1(\varphi_j)\langle\phi^*_{\lambda_k}, \varphi_i\rangle$, respectively, and $\mathbf{F}_{\lambda_k} = \{\langle F_{\lambda_k}, \varphi_j\rangle\}_{j=1}^M$ be the load vector, where $F_{\lambda_k} = \mathfrak{C}_2(F_k)\phi^*_{\lambda_k} + F_k$ and φ_j are the nodal bases of $\mathcal{V}_h$. By Algorithm $\mathcal{A}$ we find the vector $\mathbf{w}_k := \{w_j^k\}_{j=1}^M$ satisfying $(\mathbf{A}_{\lambda_k} - \mathbf{B}_{\lambda_k})\mathbf{w}_k = \mathbf{F}_{\lambda_k}$. Setting $W_{k,h} = \sum_{j=1}^M w_j^k\varphi_j$ and $\mathfrak{C}_{k,h} = \mathfrak{C}_1(W_{k,h}) + \mathfrak{C}_2(F_k)$, we obtain the approximations w_h^N and c_h^N by (1.27).



(a) $|\mathfrak{C} - \mathfrak{C}_h|$.



(b) $\|W - W_h\|_0$.



(c) $|W - W_h|_1$.

Fig. 5. Graphs of errors for $\lambda = 2\pi ki$, $k = 0, 1, 10$ and 100.

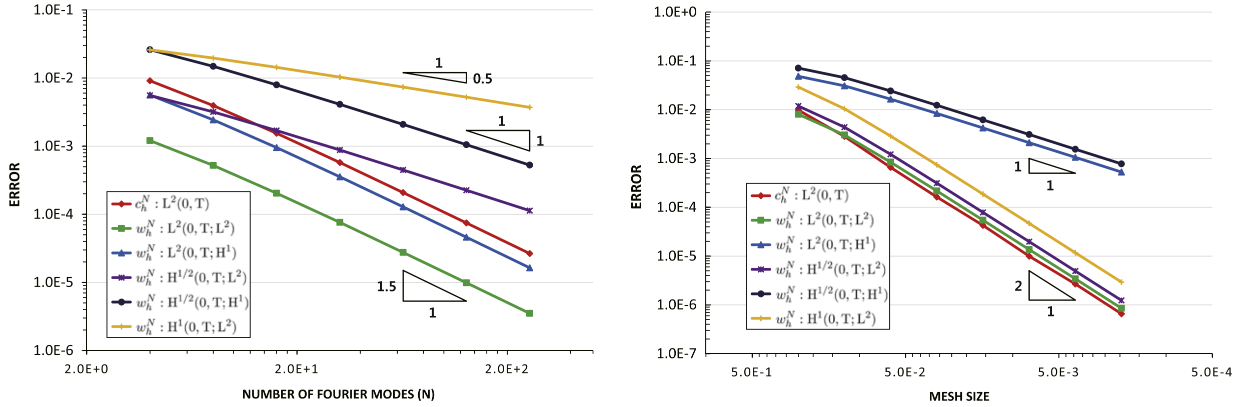
Table 2
Errors and their convergence rates for $\lambda = 2\pi i$.

h_j	$\lvert\mathfrak{C} - \mathfrak{C}_h\rvert$	Rate	$\|W - W_h\|_0$	Rate	$\lvert W - W_h\rvert_1$	Rate
2^{-1}	1.0299E−3	–	2.4220E−3	–	1.6573E−2	–
2^{-2}	2.5277E−4	2.03	8.5269E−4	1.51	1.0046E−2	0.72
2^{-3}	6.3591E−5	1.99	2.3606E−4	1.85	5.3108E−3	0.92
2^{-4}	1.4995E−5	2.08	6.0608E−5	1.96	2.6945E−3	0.98
2^{-5}	3.7546E−6	2.00	1.5254E−5	1.99	1.3523E−3	0.99
2^{-6}	8.9557E−7	2.07	3.8199E−6	2.00	6.7677E−4	1.00
2^{-7}	2.3278E−7	1.94	9.5538E−7	2.00	3.3846E−4	1.00
2^{-8}	5.7144E−8	2.03	2.3887E−7	2.00	1.6924E−4	1.00
2^{-9}	1.4187E−8	2.01	5.9718E−8	2.00	8.4622E−5	1.00
2^{-10}	3.5447E−9	2.00	1.4930E−8	2.00	4.2311E−5	1.00

Let $N_i = 2^i$. We now check convergence rates for the Heat equation. Set the following error notations:

$$\begin{aligned} \mathcal{E}(N_i, h) &= \sqrt{e(N_{i-1}, h)^2 - e(N_i, h)^2} \quad \text{for fixed } h, \\ \mathcal{E}(N, h_j) &= \sqrt{e(N, h_{j-1})^2 - e(N, h_j)^2} \quad \text{for fixed } N, \end{aligned} \tag{4.5}$$

where $e(N, h)$ denotes an appropriate error norm for the Fourier mode N and the mesh-size h.



(a) $\mathcal{E}(N_i, h_4)$, $i = 2, \ldots, 8$. (b) $\mathcal{E}(N_4, h_j)$, $j = 2, \ldots, 9$.

Fig. 6. Error graphs for the numerical example of the Heat equation.

Table 3
Numerical errors for $\|c - c_h^N\|_{L^2(0,T)}$ and $\|w - w_h^N\|_{L^2(0,T;L^2)}$.

(a) Errors for fixed $h = h_4$						
N_i	$\|c - c_{h_4}^{N_i}\|_{L^2(0,T)}$	$\mathcal{E}(N_i, h_4)$	*Rate*	$\|w - w_{h_4}^{N_i}\|_{L^2(0,T;L^2)}$	$\mathcal{E}(N_i, h_4)$	*Rate*
2^1	1.0089E−2	–	–	1.3498E−3	–	–
2^2	4.2848E−3	9.1335E−3	–	6.0759E−4	1.2053E−3	–
2^3	1.6719E−3	3.9452E−3	1.21	3.1325E−4	5.2061E−4	1.21
2^4	6.3916E−4	1.5449E−3	1.35	2.3783E−4	2.0387E−4	1.35
2^5	2.7956E−4	5.7478E−4	1.43	2.2541E−4	7.5850E−5	1.43
2^6	1.8631E−4	2.0843E−4	1.46	2.2372E−4	2.7505E−5	1.46
2^7	1.7071E−4	7.4627E−5	1.48	2.2351E−4	9.8481E−6	1.48
2^8	1.6863E−4	2.6551E−5	1.49	2.2348E−4	3.5038E−6	1.49
(b) Errors for fixed $N = N_4$						
h_j	$\|c - c_{h_j}^{N_4}\|_{L^2(0,T)}$	$\mathcal{E}(N_4, h_j)$	*Rate*	$\|w - w_{h_j}^{N_4}\|_{L^2(0,T;L^2)}$	$\mathcal{E}(N_4, h_j)$	*Rate*
2^{-1}	1.0269E−2	–	–	8.6451E−3	–	–
2^{-2}	2.9820E−3	9.8270E−3	–	3.1614E−3	8.0464E−3	–
2^{-3}	9.1650E−4	2.8377E−3	1.79	8.7280E−4	3.0385E−3	1.40
2^{-4}	6.3916E−4	6.5684E−4	2.11	2.3783E−4	8.3977E−4	1.86
2^{-5}	6.1815E−4	1.6255E−4	2.01	9.8954E−5	2.1627E−4	1.96
2^{-6}	6.1668E−4	4.2508E−5	1.94	8.2580E−5	5.4520E−5	1.99
2^{-7}	6.1660E−4	9.9721E−6	2.09	8.1446E−5	1.3639E−5	2.00
2^{-8}	6.1660E−4	2.6781E−6	1.90	8.1374E−5	3.4162E−6	2.00
2^{-9}	6.1660E−4	6.5676E−7	2.03	8.1370E−5	8.5381E−7	2.00

Consider the case $e(N, h) = \|w - w_h^N\|_{L^2(0,T;L^2)}$. By (3.19a) one sees

$$\mathcal{E}(N_i, h) = \Big[\sum_{N_{i-1}<|k|\le N_i} (\|W_k\|_0^2 - \|W_h - W_{k,h}\|_0^2)\Big]^{1/2}$$
$$\le C\Big[\sum_{N_{i-1}<|k|\le N_i} |k|^{-2}\|F_k\|_0^2\Big]^{1/2} = O(N_{i-1}^{-1}), \tag{4.6}$$

and by (3.16a) we have $\mathcal{E}(N, h_j) = O(h_{j-1}^{2-\alpha-\epsilon})$ for a very small number $\epsilon > 0$. However it is seen from Table 3(a)–(b) that the numerical errors $\mathcal{E}(N_i, h_4)$ and $\mathcal{E}(N_4, h_j)$ for the case $e(N, h) = \|w - w_h^N\|_{L^2(0,T;L^2)}$ show the (super) convergence rates 3/2 and 2, respectively. The reason is as follows: Since $\mathfrak{C}_k = e^{-\lambda_k}\lambda_k^{-3}[\lambda_k + 2 + e^{\lambda_k}(\lambda_k - 2)]$ and $\|W_k\|_0 = \frac{8}{105}|\mathfrak{C}_k| \le C|k|^{-2}$, the equality (4.6) is estimated by

$$\mathcal{E}(N_i, h) \le C\Big[\sum_{N_{i-1}<|k|\le N_i} |k|^{-4}\Big]^{1/2} \le C\Big[\int_{N_{i-1}}^{\infty} x^{-4}dx\Big]^{1/2} = O(N_{i-1}^{-3/2}).$$

Also the error $\mathcal{E}(N, h_j)$ for fixed N has the convergence rate $O(h_j^2)$ with the same reason as explained in Example 1: the error $\|W_k - W_{k,h}\|_0$ for fixed k has the rate $O(h^2)$. In Fig. 6(a) we plot the errors $\mathcal{E}(N_i, h_4)$ for some cases: $e(N, h) = \|w - w_h^N\|_{L^2(0,T;L^2)}$, $\|w - w_h^N\|_{L^2(0,T;H^1)}$, and so on, where the x-axis and the y-axis of Fig. 6(a) read the Fourier modes $N = N_i$ and the errors $\mathcal{E}(N_i, h_4)$, respectively. Similarly we plot the errors $\mathcal{E}(N_4, h_j)$ for various mesh-sizes h_j in Fig. 6(b), which also show the (super) convergence rates with the same reason explained above.

4.3. Conclusion

With several numerical examples, we have confirmed the convergence rates in some error estimates derived in Theorems 1.4 and 1.6, where Theorem 1.4 is for the finite element method, based on the corner singularity expansion, of the generalized boundary value problem with parameter on bounded plane domains with a non-convex vertex and Theorem 1.6 is for the FFEM of the initial and boundary value problems.

The prerequisite of the finite element methods used in this paper is the knowledge of the corner singularity expansion. Nevertheless the finite element solutions in numerical experiments are super-convergent with the rate 2 in L^2-error as the mesh-size varies for fixed Fourier modes. Such super-convergence results become relevant because our approach firstly provides the finite element solution for the regular part, which belongs to H^2, of the singular solution. Similar results can be found in Refs. [13,15]. Also such super-convergence provides the approximate stress intensity function with the same rate.

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References

[1] B. Heinrich, The Fourier-finite-element method for Poisson's equation in axisymmetric domains with edges, SIAM J. Numer. Anal. 33 (5) (1996) 1885–1911.
[2] B. Mercier, G. Raugel, Résolution d'un problème aux limites dans un ouvert axisymétrique par éléments finis en r, z et séries de Fourier en θ, RAIRO Anal. Numér. 16 (4) (1982) 405–461.
[3] B. Heinrich, S. Nicaise, B. Weber, Elliptic interface problems in axisymmetric domains. II. The Fourier-finite-element approximation of non-tensorial singularities, Adv. Math. Sci. Appl. 10 (2) (2000) 571–600.
[4] P. Ciarlet Jr., B. Jung, S. Kaddouri, S. Labrunie, J. Zou, The Fourier singular complement method for the Poisson problem. I. Prismatic domains, Numer. Math. 101 (3) (2005) 423–450.
[5] P. Ciarlet Jr., B. Jung, S. Kaddouri, S. Labrunie, J. Zou, The Fourier singular complement method for the Poisson problem. II. Axisymmetric domains, Numer. Math. 102 (4) (2006) 583–610.
[6] B. Heinrich, B. Jung, The Fourier-finite-element method and Nitsche mortaring, IMA J. Numer. Anal. 26 (3) (2006) 541–562.
[7] Y.P. Kim, J.R. Kweon, The Fourier-finite element method for the Poisson problem on a non-convex polyhedral cylinder, J. Comput. Appl. Math. 233 (4) (2009) 951–968.
[8] P. Chatzipantelidis, R.D. Lazarov, V. Thomée, L.B. Wahlbin, Parabolic finite element equations in nonconvex polygonal domains, BIT 46 (Suppl.) (2006) S113–S143.
[9] I. Babuška, Finite element method for domains with corners, Computing 6 (1970) 264–273.
[10] I. Babuška, R.B. Kellogg, J. Pitkäranta, Direct and inverse error estimates for finite elements with mesh refinements, Numer. Math. 33 (4) (1979) 447–471.
[11] M. Bourlard, M. Dauge, M.-S. Lubuma, S. Nicaise, Coefficients of the singularities for elliptic boundary value problems on domains with conical points. III: finite element methods on polygonal domains, SIAM J. Numer. Anal. 29 (1992) 136–155.
[12] M. Bourlard, M. Dauge, S. Nicaise, Error estimates on the coefficients obtained by the singular function method, Numer. Funct. Anal. Optim. 10 (1989) 1077–1113.
[13] S.C. Brenner, Multigrid methods for the computation of singular solutions and stress intensity factors I: corner singularities, Math. Comp. 68 (1999) 559–583.
[14] Z. Cai, S. Kim, A finite element method using singular functions for the Poisson equation: corner singularities, SIAM J. Numer. Anal. 39 (2001) 286–299.
[15] Z. Cai, S. Kim, B.C. Shin, Solution methods for the Poisson equation with corner singularities: numerical results, SIAM J. Sci. Comput. 23 (2001) 672–682.
[16] P. Chatzipantelidis, R.D. Lazarov, V. Thomée, Error estimates for a finite volume element method for parabolic equations in convex polygonal domains, Numer. Methods Partial Differential Equations 20 (2004) 650–674.
[17] P. Chatzipantelidis, R.D. Lazarov, V. Thomée, Parabolic finite volume element equations in nonconvex polygonal domains, Numer. Methods Partial Differential Equations 25 (2009) 507–525.
[18] V. Thomée, Galerkin Finite Element Methods for Parabolic Problems, Springer-Verlag, 2006.
[19] R.A. Adams, Sobolev Spaces, Academic Press, New York, 1975.
[20] P. Grisvard, Elliptic Problems in Nonsmooth Domains, Pitman, 1985.
[21] M. Dauge, Elliptic Boundary Value Problems on Corner Domains: Smoothness and Asymptotics of Solutions, Springer-Verlag, 1988.
[22] P. Grisvard, Singularities in Boundary Value Problems, Springer-Verlag, 1992.
[23] R.B. Kellogg, Notes on piecewise smooth elliptic boundary value problems. Available at: http://www.math.sc.edu/~kellogg/.
[24] C. Băcuță, V. Nistor, L.T. Zikatanov, Improving the rate of convergence of 'high order finite elements' on polygons and domains with cusps, Numer. Math. 100 (2) (2005) 165–184.
[25] H. Li, A. Mazzucato, V. Nistor, Analysis of the finite element method for transmission/mixed boundary value problems on general polygonal domains, Electron. Trans. Numer. Anal. 37 (2010) 41–69.
[26] J.R. Kweon, For the parabolic Lamé system on polygonal domains: The transport equation, J. Differential Equations 255 (2013) 1109–1131.
[27] I. Stakgold, Green's Functions and Boundary Value Problems, John Wiley & Sons, 1998.
[28] P. Grisvard, Edge behavior of the solution of an elliptic problem, Math. Nachr. 132 (1987) 281–299.
[29] P. Grisvard, Singular behavior of elliptic problems in non-hilbertian Sobolev spaces, J. Math. Pures Appl. 74 (1995) 3–33.
[30] A.H. Schatz, An observation concerning Ritz–Galerkin methods with indefinite bilinear forms, Math. Comp. 28 (1974) 959–962.
[31] H.J. Choi, J.R. Kweon, The stationary Navier–Stokes system with no-slip boundary condition on polygons: corner singularity and regularity, Comm. Partial Differential Equations 38 (2013) 1235–1255.
[32] P. Clément, Approximation by finite element functions using local regularization, RAIRO Modél. Math. Anal. Numér. R-2 (1975) 77–84.
[33] P.G. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland, 1978.
[34] M. Feistauer, On the finite element approximation of functions with noninteger derivatives, Numer. Funct. Anal. Optim. 10 (1989) 91–110.
[35] J.R. Kweon, An optimal order convergence for a weak formulation of the compressible Stokes system with inflow boundary condition, Numer. Math. 86 (2000) 305–318.