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On Teitelbaum type $\mathcal{L}$-invariants of Hilbert modular forms attached to definite quaternions

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We generalize Teitelbaum's work on the definition of the $\mathcal{L}$-invariant to Hilbert modular forms that arise from definite quaternion algebras over totally real fields by the Jacquet–Langlands correspondence. Conjecturally this coincides with the Fontaine–Mazur type $\mathcal{L}$-invariant, defined by applying Fontaine's theory to the Galois representation associated to Hilbert modular forms. An exceptional zero conjecture for the p-adic L-function of Hilbert modular forms is also proposed.

Contents

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1. Introduction

We generalize Teitelbaum's work [16] on the definition of the $\mathcal{L}$-invariant to Hilbert modular forms that arise from definite quaternion algebras over totally real fields by the Jacquet–Langlands correspondence.

More precisely, suppose that $\mathbf{f}$ is a Hilbert eigenform over a totally real field F (see Section 2.5) that is special at a prime $\mathfrak{p}|p$ of F (in the sense that the $\mathfrak{p}$-component of the cuspidal automorphic representation generated by $\mathbf{f}$ is the special representation of $\mathrm{GL}_2(F_\mathfrak{p})$), and that $\mathbf{f}$ arises from a totally definite quaternion algebra B over F by the Jacquet–Langlands correspondence. Assume that B splits at $\mathfrak{p}$, and that $\mathfrak{l}$ exactly divides the conductor of $\mathbf{f}$ if B is ramified at $\mathfrak{l}$. Then we can define the $\mathcal{L}$-invariant of $\mathbf{f}$ at the prime $\mathfrak{p}$ in the style of Teitelbaum (Definition 3.4). One new feature that arises in the totally real case is that we have an $\mathcal{L}$-invariant $\mathcal{L}_\mathfrak{p}^{\sigma,\mathrm{Tei}}(\mathbf{f})$ for each embedding σ of $F_\mathfrak{p}/\mathbf{Q}_p$ into $\overline{\mathbf{Q}}_p$.

As in Teitelbaum's case ($F = \mathbf{Q}$), we need a pair of group cocycles associated to $\mathbf{f}$ in order to define the $\mathcal{L}$-invariant. One of them is in the manner of Schneider as in [16], and the other is defined by using Coleman integrals given by the periods of the rigid analytic modular form associated to the Jacquet–Langlands correspondence of $\mathbf{f}$. Another new feature that occurs when $F \neq \mathbf{Q}$ is that the rigid analytic modular forms involved are in general vector valued (rather than scalar valued).

In the case $F = \mathbf{Q}$ the Fontaine–Mazur $\mathcal{L}$-invariant [10] is defined by applying Fontaine's theory to the Galois representation associated to the eigenform, and the definition can be generalized to the Hilbert modular case (which will be explained in 3.2). Then we conjecture that the Teitelbaum type $\mathcal{L}$-invariant coincides with the corresponding Fontaine–Mazur type $\mathcal{L}$-invariant. In the case $F = \mathbf{Q}$ this was proved in [8], by making use of an explicit version of the comparison theorem in p-adic Hodge theory.

The organization of this paper is as follows. We collect preliminaries about automorphic forms on totally definite quaternion algebras in Section 2. The $\mathcal{L}$-invariant in the style of Teitelbaum is defined in Section 3. Then we state the exceptional zero conjecture in Section 4.

Notation 1.1. Throughout the paper F is a totally real field, with ring of integers $\mathcal{O}_F$. Denote by $d = [F : \mathbf{Q}]$ the degree of F over $\mathbf{Q}$. The ring of adeles, and finite adeles will be denoted as $\mathbf{A}_F$ and $\widehat{F}$.

Fix a rational prime p. For each $\mathfrak{p}|p$ let $F_\mathfrak{p}$ be the completion of F at $\mathfrak{p}$. Thus $F_p := F \otimes_\mathbf{Q} \mathbf{Q}_p = \prod_{\mathfrak{p}|p} F_\mathfrak{p}$.

For each prime $\mathfrak{l}$ of F, denote by $\mathrm{val}_\mathfrak{l}$ the normalized valuation of $\mathcal{O}_{F_\mathfrak{l}}$, whose value on the uniformizer of $\mathcal{O}_{F_\mathfrak{l}}$ is one. Put $|.|_\mathfrak{l}$ to be the normalized absolute value of $F_\mathfrak{l}$ given by $|x|_\mathfrak{l} = \mathcal{N}\mathfrak{l}^{-\mathrm{val}_\mathfrak{l} x}$. Here in general we will denote by $\mathcal{N}$ the norm operation from F to $\mathbf{Q}$, either over the field itself, their completion, the adeles, or at the level of ideal (the subscripts that occur would indicate the field extensions involved).

Let $\chi_{\mathbf{Q},\mathrm{cycl}}$ be the p-adic cyclotomic character of $\mathbf{Q}$, which by class field theory is regarded as a Hecke character $\chi_{\mathbf{Q},\mathrm{cycl}} : \mathbf{A}_\mathbf{Q}^\times/\mathbf{Q}^\times \to \mathbf{Z}_p^\times$. The class field theory isomorphism is normalized so that $\chi_{\mathbf{Q},\mathrm{cycl}}(z) = z$ for $z \in \mathbf{Z}_p^\times$. Let $\chi_{F,\mathrm{cycl}} : \mathbf{A}_F^\times/F^\times \to \mathbf{Z}_p^\times$ be the Hecke character obtained by composing $\chi_{\mathbf{Q},\mathrm{cycl}}$ with the norm map from $\mathbf{A}_F^\times$ to $\mathbf{A}_\mathbf{Q}^\times$. The character $\chi_{F,\mathrm{cycl}}$ is trivial on the archimedean connected component of $\mathbf{A}_F^\times$, hence we can view $\chi_{F,\mathrm{cycl}}$ as a character on $\widehat{F}^\times/F_+^\times$, where $F_+^\times$ is the set of totally positive elements of F.

Fix once and for all an embedding of $\overline{\mathbf{Q}}$ into $\overline{\mathbf{Q}}_p$ and $\mathbf{C}$.

Denote by $|.|_p$ the absolute value on $\overline{\mathbf{Q}}_p$ normalized by the condition $|p|_p = 1/p$. As usual $\mathbf{C}_p$ is the completion of $\overline{\mathbf{Q}}_p$ with respect to $|.|_p$. Denote by ord_p the valuation of $\mathbf{C}_p^\times$, normalized by the condition $\mathrm{ord}_p(p) = 1$.

Denote by I the set of embeddings of F into $\overline{\mathbf{Q}}_p$. We denote by $\mathbf{Z}[I]$ the free abelian group generated by I. Note that we can partition $I = \bigsqcup_{\mathfrak{p}|p} I_\mathfrak{p}$, where $I_\mathfrak{p}$ consists of those embeddings that factor through $F_\mathfrak{p}$. We have $\#I_\mathfrak{p} = [F_\mathfrak{p} : \mathbf{Q}_p]$.

Finally for any ring A, we denote by $M_2(A)$ the ring of 2×2 matrices with coefficients in A.

2. Preliminaries

2.1. Automorphic forms on totally definite quaternion algebras

Let B be a totally definite quaternion algebra over F, i.e. $B \otimes_{F,\nu} \mathbf{R}$ is isomorphic to Hamilton's quaternions, for all the real embeddings $\nu : F \to \mathbf{R}$. We refer to Vigneras' book [18] for the theory of quaternion algebras.

Denote by $\mathfrak{n}^-$ the discriminant of B. Thus $\mathfrak{n}^-$ is the square-free ideal of $\mathcal{O}_F$ that is equal to the product of the set of finite primes of $\mathcal{O}_F$ at which B ramifies. Note that the number of prime factors of $\mathfrak{n}^-$ is congruent to $[F : \mathbf{Q}]$ mod 2. We denote by J the set of primes of $\mathcal{O}_F$ above p that do not divide $\mathfrak{n}^-$. The reader is advised to take J to be the set of all primes above p, i.e. that $\mathfrak{n}^-$ is relatively prime to $p\mathcal{O}_F$, on first reading.

We now define automorphic forms on $B^\times$. Let $\widehat{B}^\times = (B \otimes_F \widehat{F})^\times$ be the group of finite adelic points of $B^\times$. Given $b \in \widehat{B}^\times$, and a place ν of F, we will denote by b_ν the component of b at ν. We will generally identify the finite places of F with prime ideals of $\mathcal{O}_F$, so if ν corresponds to a prime $\mathfrak{l}$, then we will also write $b_\mathfrak{l}$ for the corresponding component of b. On the other hand, we will write $b^\mathfrak{l}$ for the element of $\widehat{B}^\times$ obtained from b by replacing $b_\mathfrak{l}$ by the identity. Finally, we will write $b_p \in \prod_{\mathfrak{q}|p} B_\mathfrak{q}^\times$ for the element $(b_\mathfrak{q})_{\mathfrak{q}|p}$.

We fix an isomorphism

$$B \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}_p \cong \prod_{\sigma \in I} M_2(\overline{\mathbf{Q}}_p)$$

which is equivalent to the data: for all $\mathfrak{q}|p$, and $\sigma \in I_{\mathfrak{q}}$, an isomorphism

$$B_{\mathfrak{q}} \otimes_{F_{\mathfrak{q}},\sigma} \overline{\mathbf{Q}}_p \cong M_2(\overline{\mathbf{Q}}_p) \tag{2.1}$$

(here $B_{\mathfrak{q}} = B \otimes_F F_{\mathfrak{q}}$).

For each prime $\mathfrak{l}$ not dividing $\mathfrak{n}^-$, fix an isomorphism of $F_{\mathfrak{l}}$-algebras:

$$\iota_{\mathfrak{l}} : B_{\mathfrak{l}} = B \otimes_F F_{\mathfrak{l}} \to M_2(F_{\mathfrak{l}}) \tag{2.2}$$

which induces an isomorphism of $B_{\mathfrak{l}}^\times$ and $\mathrm{GL}_2(F_{\mathfrak{l}})$. If $\mathfrak{l} = \mathfrak{q} \in J$, we assume that the isomorphism (2.1) for each $\sigma \in I_{\mathfrak{q}}$ is induced from that of (2.2).

Let $\Sigma = \prod_{\mathfrak{l}} \Sigma_{\mathfrak{l}}$ be an open compact subgroup of $\widehat{B}^\times$. Assume that the image of $\Sigma_{\mathfrak{q}}$ under $\iota_{\mathfrak{q}}$ is contained in $\mathrm{GL}_2(\mathcal{O}_{F_{\mathfrak{q}}})$ for all $\mathfrak{q} \in J$. In the sequel, $\Sigma_{\mathfrak{q}}$ would then be identified as subgroup of $\mathrm{GL}_2(\mathcal{O}_{F_{\mathfrak{q}}})$ for each $\mathfrak{q} \in J$.

Definition 2.1. For an embedding $\sigma \in I_{\mathfrak{q}}$, and integers n, v with $n \geq 0$, let $L_\sigma(n, v)$ be the $\mathbf{C}_p$-vector space of polynomials in one variable of degree at most n, with coefficients in $\mathbf{C}_p$, and with the right action of $B_{\mathfrak{q}}^\times$ on $L_\sigma(n, v)$ given as follows: for $\gamma \in B_{\mathfrak{q}}^\times$, write $\gamma^\sigma = \left(\begin{smallmatrix} a^\sigma & b^\sigma \\ c^\sigma & d^\sigma \end{smallmatrix}\right)$ to be the image of γ under

$$B_{\mathfrak{q}}^\times \to (B_{\mathfrak{q}} \otimes_{F_{\mathfrak{q}},\sigma} \overline{\mathbf{Q}}_p)^\times \overset{(2.1)}{\cong} \mathrm{GL}_2(\overline{\mathbf{Q}}_p),$$

then we define

$$(P|\gamma)(T) = \det\left(\gamma^\sigma\right)^v \left(c^\sigma T + d^\sigma\right)^n P\left(\frac{a^\sigma T + b^\sigma}{c^\sigma T + d^\sigma}\right). \tag{2.3}$$

For a pair of vectors $\underline{n} = (n_\sigma)_{\sigma \in I}$, $\underline{v} = (v_\sigma)_{\sigma \in I} \in \mathbf{Z}[I]$, with $n_\sigma \geq 0$ for all $\sigma \in I$, put

$$L(\underline{n}, \underline{v}) = \bigotimes_{\sigma \in I} L_\sigma(n_\sigma, v_\sigma) = \bigotimes_{\mathfrak{q}|p} \bigotimes_{\sigma \in I_{\mathfrak{q}}} L_\sigma(n_\sigma, v_\sigma).$$

For each $\mathfrak{q}|p$ we have the natural tensor product right action of $B_{\mathfrak{q}}^\times$ on $\bigotimes_{\sigma \in I_{\mathfrak{q}}} L_\sigma(n_\sigma, v_\sigma)$, hence the product right-action of $B_p^\times = \prod_{\mathfrak{q}|p} B_{\mathfrak{q}}^\times$ on $L(\underline{n}, \underline{v})$. Define $V_\sigma(n_\sigma, v_\sigma)$ to be the $\mathbf{C}_p$-dual of $L_\sigma(n_\sigma, v_\sigma)$, and $V(\underline{n}, \underline{v})$ to be the $\mathbf{C}_p$-dual of $L(\underline{n}, \underline{v})$, with the dual left action of $B_p^\times$ which is given by $\langle P|\gamma, \Phi\rangle = \langle P, \gamma \cdot \Phi\rangle$ for $\gamma \in B_p^\times$, $P \in L_\sigma(n, v)$ and $\Phi \in V_\sigma(n, v)$.

Let $\underline{t} = (1, \cdots, 1) \in \mathbf{Z}[I]$. Suppose that there is an integer m such that the condition

$$\underline{n} + 2\underline{v} = m\underline{t} \tag{2.4}$$

is satisfied. Note that for a given $\underline{n}$, the set of vectors $\underline{v}$ that satisfies condition (2.4) (for variable integer m) differs from each other by integer multiples of $\underline{t}$.

Definition 2.2. With the above notations, a p-adic automorphic form on $B^\times$ of weight $(\underline{n}, \underline{v})$, level Σ, is a function

$$\Phi : \widehat{B}^\times \to V(\underline{n}, \underline{v})$$

that satisfies:

$$\Phi(z\gamma bu) = \chi_{F,\mathrm{cycl}}^{-m}(z)\bigl(u_p^{-1} \cdot \Phi(b)\bigr) \tag{2.5}$$

for all $\gamma \in B^\times$, $b \in \widehat{B}^\times$, $u \in \Sigma$ and $z \in \widehat{F}^\times$. Denote by $S^B_{\underline{n},\underline{v}}(\Sigma)$ the space of such forms.

Note that a form Φ of level Σ is determined by its values on a set of representatives of the double coset space

$$B^\times \backslash \widehat{B}^\times / \Sigma$$

which is finite (being both compact and discrete).

If $S^B_{\underline{n},\underline{v}}(\Sigma)$ is non-zero, then by (2.5) the character $\chi_{F,\mathrm{cycl}}^{-m}$ has to factor through $\widehat{F}^\times / F^\times$, which implies that m is even, and hence n_σ is even for all $\sigma \in I$.

Remark 2.3. Let $\Phi \in S^B_{\underline{n},\underline{v}}(\Sigma)$. For any $\mathfrak{q}|p$, let $\pi_\mathfrak{q}$ be a uniformizer of $F_\mathfrak{q}$, identified as the idele that is equal to $\pi_\mathfrak{q}$ at the place $\mathfrak{q}$ and 1 elsewhere. Applying Eq. (2.5) with $z = \pi_\mathfrak{q}$, we have

$$\begin{aligned}\Phi(\pi_\mathfrak{q} b) &= \chi_{F,\mathrm{cycl}}(\pi_\mathfrak{q})^{-m}\Phi(b)\\ &= \bigl|\mathcal{N}_{F_\mathfrak{q}/\mathbf{Q}_p}(\pi_\mathfrak{q})\bigr|_p^{-m}\mathcal{N}_{F_\mathfrak{q}/\mathbf{Q}_p}(\pi_\mathfrak{q})^{-m}\Phi(b) \quad \text{for all } b \in \widehat{B}^\times.\end{aligned} \tag{2.6}$$

Remark 2.4. Given $\Phi \in S^B_{\underline{n},\underline{v}}(\Sigma)$, and an integer r, define Φ' by

$$\Phi'(g) = \chi_{F,\mathrm{cycl}}(\mathrm{Nrd}_{\widehat{B}/\widehat{F}}\, g)^{-r}\Phi(g)$$

where $\mathrm{Nrd}_{\widehat{B}/\widehat{F}} : \widehat{B} \to \widehat{F}$ is the map induced by the reduced norm map $\mathrm{Nrd}_{B/F}$ from B to F. Then $\Phi' \in S^B_{\underline{n},\underline{v}'}(\Sigma)$, where

$$\underline{v}' = \underline{v} + r\underline{t}$$

(and hence $\underline{n} + 2\underline{v}' = m'\underline{t}$, with $m' = m + 2r$). Here we are identifying the underlying vector space of $V(\underline{n}, \underline{v})$ and $V(\underline{n}, \underline{v}')$. It follows that $S^B_{\underline{n},\underline{v}}(\Sigma) \cong S^B_{\underline{n},\underline{v}'}(\Sigma)$ via this twisting operation.

In the case where $\underline{n} = \underline{0}$ (and hence $\underline{v} = \frac{m}{2}\underline{t}$) one usually considers the space $S_{\underline{0},\underline{v}}(\Sigma)$ modulo the $\mathbf{C}_p$-span of the form Φ_0 given by $\Phi_0 = (\chi_{F,\mathrm{cycl}} \circ \mathrm{Nrd}_{\widehat{B}/\widehat{F}})^{-m/2}$, as this does not correspond to cusp form under the Jacquet–Langlands correspondence (see Section 2.5 below).

2.2. Hecke operators

Recall the definition of Hecke operators. For each prime $\mathfrak{l} \nmid \mathfrak{n}^-$ at which $\Sigma_\mathfrak{l}$ is maximal, one can define the Hecke operators $T_\mathfrak{l}$ as follows. Fix the isomorphism $\iota_\mathfrak{l} : B_\mathfrak{l} \to M_2(F_\mathfrak{l})$ such that $\Sigma_\mathfrak{l}$ becomes identified as $\mathrm{GL}_2(\mathcal{O}_{F_\mathfrak{l}})$. Let $\pi_\mathfrak{l}$ be a uniformizer of $\mathcal{O}_{F_\mathfrak{l}}$, and let $k_\mathfrak{l}$ be the residue field at $\mathfrak{l}$. Given a double coset decomposition:

$$\mathrm{GL}_2(\mathcal{O}_{F_\mathfrak{l}}) \begin{pmatrix} 1 & 0 \\ 0 & \pi_\mathfrak{l} \end{pmatrix} \mathrm{GL}_2(\mathcal{O}_{F_\mathfrak{l}}) = \bigsqcup_{r \in \mathbf{P}^1(k_\mathfrak{l})} \sigma_r(\mathfrak{l})\, \mathrm{GL}_2(\mathcal{O}_{F_\mathfrak{l}}). \tag{2.7}$$

Define the action of the Hecke operator $T_\mathfrak{l}$ on $S^B_{\underline{n},\underline{v}}(\Sigma)$ by the rule:

$$(T_\mathfrak{l}\Phi)(b) = \begin{cases} \sum_{r \in \mathbf{P}^1(k_\mathfrak{l})} \Phi(b \cdot \sigma_r(\mathfrak{l})) & \text{if } \mathfrak{l} \nmid p, \\ \sum_{r \in \mathbf{P}^1(k_\mathfrak{l})} \sigma_r(\mathfrak{l}) \cdot \Phi(b \cdot \sigma_r(\mathfrak{l})) & \text{if } \mathfrak{l} | p. \end{cases} \tag{2.8}$$

It is clear that this is independent of the choice of the $\sigma_r(\mathfrak{l})$.

Suppose that for $\mathfrak{l} \nmid \mathfrak{n}^-$ the level $\Sigma_\mathfrak{l}$ is not maximal, but is an Iwahori subgroup of $\mathrm{GL}_2(\mathcal{O}_{F_\mathfrak{l}})$. Then we can define the operators $U_\mathfrak{l}$. To define it, first recall the definition of Iwahori subgroups.

In general for any $\mathfrak{l}$, let $\pi_\mathfrak{l}$ be a uniformizer of $\mathcal{O}_{F_\mathfrak{l}}$ as above. Then for $m \geq 1$, we define the Iwahori subgroup $I_{\mathfrak{l}^m}$ of $\mathrm{GL}_2(\mathcal{O}_{F_\mathfrak{l}})$ of level $\mathfrak{l}^m$, by

$$I_{\mathfrak{l}^m} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_{F_\mathfrak{l}}) \;\middle|\; c \equiv 0 \bmod \pi_\mathfrak{l}^m \right\}.$$

Similarly put

$$M_2(\mathfrak{l}^m; \mathcal{O}_{F_\mathfrak{l}}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{O}_{F_\mathfrak{l}}) \;\middle|\; c \equiv 0 \bmod \pi_\mathfrak{l}^m \right\}.$$

Suppose that Σ is a level, and $\mathfrak{l} \nmid \mathfrak{n}^-$, such that $\Sigma_\mathfrak{l} = I_{\mathfrak{l}^n}$ for some $n \geq 1$. Given a double coset decomposition

$$I_{\mathfrak{l}^n} \begin{pmatrix} 1 & 0 \\ 0 & \pi_\mathfrak{l} \end{pmatrix} I_{\mathfrak{l}^n} = \bigsqcup_{r \in k_\mathfrak{l}} \widehat{\sigma}_r(\mathfrak{l}) I_{\mathfrak{l}^n}, \tag{2.9}$$

define the action of the Hecke operator $U_\mathfrak{l}$ on $S^B_{\underline{n},\underline{v}}(\Sigma)$ by the rule

$$(U_\mathfrak{l}\Phi)(b) = \begin{cases} \sum_{r \in k_\mathfrak{l}} \Phi(b \cdot \widehat{\sigma}_r(\mathfrak{l})) & \text{if } \mathfrak{l} \nmid p, \\ \sum_{r \in k_\mathfrak{l}} \widehat{\sigma}_r(\mathfrak{l}) \cdot \Phi(b \cdot \widehat{\sigma}_r(\mathfrak{l})) & \text{if } \mathfrak{l} | p. \end{cases} \tag{2.10}$$

One can take for example:

$$\widehat{\sigma}_r(\mathfrak{l}) = \begin{pmatrix} 1 & 0 \\ \widetilde{r}\pi_{\mathfrak{l}}^{n} & \pi_{\mathfrak{l}} \end{pmatrix} \tag{2.11}$$

where $\widetilde{r} \in \mathcal{O}_{\mathfrak{l}}$ maps to r.

2.3. *Choice of levels*

In this paper, the level Σ is defined by the groups of units of local Eichler orders of B. Thus let $\mathfrak{a}$ be an ideal of $\mathcal{O}_F$, relatively prime to $\mathfrak{n}^-$. For any prime $\mathfrak{l}$, let $R_{\mathfrak{l}}$ be a local order of $B_{\mathfrak{l}}$ satisfying the condition:

$$R_{\mathfrak{l}} = \text{the (unique) maximal order of } B_{\mathfrak{l}} \text{ if } \mathfrak{l} \text{ divides } \mathfrak{n}^-$$

and

$$R_{\mathfrak{l}} = \text{an Eichler order of level } \mathfrak{l}^{\mathrm{val}_{\mathfrak{l}}(\mathfrak{a})} \text{ if } \mathfrak{l} \text{ is prime to } \mathfrak{n}^-.$$

For $\mathfrak{l}$ not dividing $\mathfrak{n}^-$, we will assume that under the isomorphism $\iota_{\mathfrak{l}} : B_{\mathfrak{l}} \to M_2(F_{\mathfrak{l}})$, the image of $R_{\mathfrak{l}}$ is $M_2(\mathfrak{l}^{\mathrm{val}_{\mathfrak{l}}\mathfrak{a}}; \mathcal{O}_{F_{\mathfrak{l}}})$. Thus we have $\iota_{\mathfrak{l}}(R_{\mathfrak{l}}^{\times}) = I_{\mathfrak{l}^{\mathrm{val}_{\mathfrak{l}}(\mathfrak{a})}}$.

Let $\widehat{R} = \prod_{\mathfrak{l}} R_{\mathfrak{l}}$. Then $R := B \cap \widehat{R}$ is an Eichler order of B of level $\mathfrak{a}$. We will denote by $\Sigma(\mathfrak{a}, \mathfrak{n}^-)$ the level given by $\widehat{R}^{\times}$ for the above choices of the local orders $R_{\mathfrak{l}}$.

Notation 2.5. We will write $S^B_{\underline{n},\underline{v}}(\mathfrak{a}, \mathfrak{n}^-)$ for $S^B_{\underline{n},\underline{v}}(\Sigma(\mathfrak{a}, \mathfrak{n}^-))$.

For the levels of the type $\Sigma = \Sigma(\mathfrak{a}, \mathfrak{n}^-)$ one can also define the operators $U_{\mathfrak{l}}$ for $\mathfrak{l}|\mathfrak{n}^-$ as in (2.10) by using the double coset $\Sigma_{\mathfrak{l}}\omega_{\mathfrak{l}}\Sigma_{\mathfrak{l}} = \omega_{\mathfrak{l}}\Sigma_{\mathfrak{l}}$, where $\omega_{\mathfrak{l}}$ is a uniformizer of the maximal order $R_{\mathfrak{l}}$ of $B_{\mathfrak{l}}$ (note that $\Sigma_{\mathfrak{l}} = R_{\mathfrak{l}}^{\times}$), i.e. if $\mathfrak{l}|\mathfrak{n}^-$, then

$$(U_{\mathfrak{l}}\Phi)(b) = \begin{cases} \Phi(b\omega_{\mathfrak{l}}) & \text{if } \mathfrak{l} \nmid p, \\ \omega_{\mathfrak{l}} \cdot \Phi(b\omega_{\mathfrak{l}}) & \text{if } \mathfrak{l}|p. \end{cases}$$

In the rest of the paper, we will write the ideal $\mathfrak{a}$ that occurs in the level in the form

$$\mathfrak{a} = \mathfrak{m}\mathfrak{n}^+$$

with $\mathfrak{n}^+$ relatively prime to $p\mathcal{O}_F$, and $\mathfrak{m}$ divisible only by primes above p.

Put $\mathfrak{n} = \mathfrak{m}\mathfrak{n}^+\mathfrak{n}^-$. We denote by $\mathbf{T}$ the polynomial algebra over $\mathbf{Z}$ generated by the symbols $T_{\mathfrak{l}}$ for $\mathfrak{l} \nmid \mathfrak{n}$, and the symbols $U_{\mathfrak{l}}$ for $\mathfrak{l}|\mathfrak{n}$. The algebra $\mathbf{T}$ acts on the space of automorphic forms, and other objects (cohomology groups for instance, see Section 2.5). A form $\Phi \in S^B_{\underline{n},\underline{v}}(\mathfrak{m}\mathfrak{n}^+, \mathfrak{n}^-)$ is called an eigenform if it is an eigenvector for the action of $\mathbf{T}$.

Suppose now that $\mathfrak{l}$ is a prime that divides $\mathfrak{m}\mathfrak{n}^+$. Define the trace operator

$$\mathrm{Tr}^{\mathfrak{m}\mathfrak{n}^+}_{\mathfrak{m}\mathfrak{n}^+/\mathfrak{l}} : S^B_{\underline{n},\underline{v}}\big(\mathfrak{m}\mathfrak{n}^+, \mathfrak{n}^-\big) \to S^B_{\underline{n},\underline{v}}\left(\frac{\mathfrak{m}\mathfrak{n}^+}{\mathfrak{l}}, \mathfrak{n}^-\right)$$

as follows: given a form $\Phi \in S^B_{\underline{n},\underline{v}}(\mathfrak{m}\mathfrak{n}^+, \mathfrak{n}^-)$,

$$\mathrm{Tr}^{\mathfrak{m}\mathfrak{n}^+}_{\mathfrak{m}\mathfrak{n}^+/\mathfrak{l}}(\Phi)(g) = \begin{cases} \sum_r \Phi(g\tau_r) & \text{if } \mathfrak{l} \nmid p, \\ \sum_r \tau_r \cdot \Phi(g\tau_r) & \text{if } \mathfrak{l} | p. \end{cases}$$

Here $\{\tau_r\}$ run over a set of left coset representatives of $I_{\mathfrak{l}^{n-1}}$ modulo $I_{\mathfrak{l}^n}$ where $n = \mathrm{val}_{\mathfrak{l}}(\mathfrak{m}\mathfrak{n}^+)$ (if $n = 1$ then $I_{\mathfrak{l}^{n-1}}$ is interpreted as $\mathrm{GL}_2(\mathcal{O}_{F_{\mathfrak{l}}})$). For example if $\mathfrak{l}$ divides $\mathfrak{m}\mathfrak{n}^+$ exactly (i.e. $n = 1$) then one can take (with the r indexed by $\mathbf{P}^1(k_{\mathfrak{l}})$):

$$\tau_r = \begin{pmatrix} 1 & 0 \\ \widetilde{r} & 1 \end{pmatrix} \quad \text{for } r \in k_{\mathfrak{l}} \tag{2.12}$$

and

$$\tau_\infty = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

A form $\Phi \in S^B_{\underline{n},\underline{v}}(\mathfrak{m}\mathfrak{n}^+, \mathfrak{n}^-)$ is said to be *new* at $\mathfrak{l}$, if

$$\mathrm{Tr}^{\mathfrak{m}\mathfrak{n}^+}_{\mathfrak{m}\mathfrak{n}^+/\mathfrak{l}}(\Phi) = 0.$$

An eigenform $\Phi \in S^B_{\underline{n},\underline{v}}(\mathfrak{m}\mathfrak{n}^+, \mathfrak{n}^-)$ is called a newform if it is new at all primes dividing $\mathfrak{m}\mathfrak{n}^+$.

Finally define the Atkin–Lehner operator at $\mathfrak{l}$ as:

$$\begin{aligned} W_{\mathfrak{l}} : S^B_{\underline{n},\underline{v}}\big(\mathfrak{m}\mathfrak{n}^+, \mathfrak{n}^-\big) &\to S^B_{\underline{n},\underline{v}}\big(\mathfrak{m}\mathfrak{n}^+, \mathfrak{n}^-\big) \\ W_{\mathfrak{l}}(\Phi)(g) &= b_{\mathfrak{l}} \cdot \Phi(gb_{\mathfrak{l}}), \end{aligned} \tag{2.13}$$

here $b_{\mathfrak{l}} = \left(\begin{smallmatrix} 0 & -1 \\ \pi_{\mathfrak{l}} & 0 \end{smallmatrix}\right)$. In particular if the prime $\mathfrak{l}$ does not lie above p, then we have $W_{\mathfrak{l}}(\Phi)(g) = \Phi(gb_{\mathfrak{l}})$. An immediate calculation shows that when $\mathfrak{l}$ divides $\mathfrak{m}\mathfrak{n}^+$ exactly, a form $\Phi \in S^B_{\underline{n},\underline{v}}(\mathfrak{m}\mathfrak{n}^+, \mathfrak{n}^-)$ is new at $\mathfrak{l}$ if and only if

$$U_{\mathfrak{l}}(\Phi) = -W_{\mathfrak{l}}(\Phi). \tag{2.14}$$

2.4. Harmonic cocycles on Bruhat–Tits tree

As in the previous section we write $\Sigma = \Sigma(\mathfrak{m}\mathfrak{n}^+, \mathfrak{n}^-)$. Fix a $\mathfrak{p} \in J$. Identify $B^\times_{\mathfrak{p}}$ with $\mathrm{GL}_2(F_{\mathfrak{p}})$ via $\iota_{\mathfrak{p}}$.

Let $\{t_{i,\mathfrak{p}}\}_{i=1}^{h_\mathfrak{p}}$ be a set of representatives of

$$F_+^\times \backslash \widehat{F}^\times / \widehat{\mathcal{O}}_F^\times F_\mathfrak{p}^\times$$

(which is equal to the quotient of the strict ideal class group of F by the image of $F_\mathfrak{p}^\times$ and $h_\mathfrak{p}$ is the order of this group). We assume that the $t_{i,\mathfrak{p}}$'s are chosen to have trivial components at all $\mathfrak{q}|p$. Fix $x_{i,\mathfrak{p}} \in \widehat{B}^\times$ with $(x_{i,\mathfrak{p}})_\mathfrak{q} = 1$ for all $\mathfrak{q}|p$, such that $\mathrm{Nrd}_{\widehat{B}/\widehat{F}}(x_{i,\mathfrak{p}}) = t_{i,\mathfrak{p}}$ for $i = 1, \cdots, h_\mathfrak{p}$. The theorem of the norm and the strong approximation theorem (Theorems 4.1 and 4.3 of [18], Chapter 3) give a decomposition:

$$\widehat{B}^\times = \bigsqcup_{i=1}^{h_\mathfrak{p}} B^\times x_{i,\mathfrak{p}} \,\mathrm{GL}_2(F_\mathfrak{p})\Sigma. \tag{2.15}$$

More precisely if $y \in \widehat{B}^\times$, then the unique index i of (2.15) to which y belongs is determined by the condition that the class of $\mathrm{Nrd}_{\widehat{B}/\widehat{F}}(yx_{i,\mathfrak{p}}^{-1})$ in $F_+^\times \backslash \widehat{F}^\times / \widehat{\mathcal{O}}_F^\times F_\mathfrak{p}^\times$ is trivial.

Define, for $i = 1, \cdots, h_\mathfrak{p}$:

$$\widetilde{\Gamma}_i^\mathfrak{p} = \widetilde{\Gamma}_i^\mathfrak{p}(\mathfrak{m}\mathfrak{n}^+, \mathfrak{n}^-) := \{\gamma \in B^\times \mid \gamma_\mathfrak{l} \in (x_{i,\mathfrak{p}})_\mathfrak{l}\Sigma_\mathfrak{l}(x_{i,\mathfrak{p}})_\mathfrak{l}^{-1} \text{ for } \mathfrak{l} \neq \mathfrak{p}\}.$$

Using (2.15), we have a bijection:

$$\bigsqcup_{i=1}^{h_\mathfrak{p}} \widetilde{\Gamma}_i^\mathfrak{p} \backslash \mathrm{GL}_2(F_\mathfrak{p})/\Sigma_\mathfrak{p} \xrightarrow{\sim} B^\times \backslash \widehat{B}^\times / \Sigma \tag{2.16}$$

where for $g \in \mathrm{GL}_2(F_\mathfrak{p})$, the class of g in $\widetilde{\Gamma}_i^\mathfrak{p} \backslash \mathrm{GL}_2(F_\mathfrak{p})/\Sigma_\mathfrak{p}$ gets mapped to the class of $x_{i,\mathfrak{p}} \cdot g$ in $B^\times \backslash \widehat{B}^\times / \Sigma$, with g regarded as the element of $\widehat{B}^\times$ that is equal to g at the place $\mathfrak{p}$, and equal to identity at other places.

Using (2.16), we see that a form $\Phi \in S^B_{\underline{n},\underline{v}}(\mathfrak{m}\mathfrak{n}^+, \mathfrak{n}^-)$, with $\underline{n}+2\underline{v} = m\underline{t}$, can be identified as an $h_\mathfrak{p}$-tuples of function $\phi_\mathfrak{p}^1, \cdots, \phi_\mathfrak{p}^{h_\mathfrak{p}}$ on $\mathrm{GL}_2(F_\mathfrak{p})$, by the rule: $\phi_\mathfrak{p}^i(g) = \Phi(x_{i,\mathfrak{p}} \cdot g)$, for $i = 1, \cdots, h_\mathfrak{p}$. The functions $\phi_\mathfrak{p}^i$, satisfy:

$$\phi_\mathfrak{p}^i(\gamma_\mathfrak{p} guz) = \chi_{F,\mathrm{cycl}}^{-m}(z)\bigl(u^{-1}\gamma_p^\mathfrak{p} \cdot \phi_\mathfrak{p}^i(g)\bigr) \quad \text{for } \gamma \in \widetilde{\Gamma}_i^\mathfrak{p},\ g \in \mathrm{GL}_2(F_\mathfrak{p}),\ u \in \Sigma_\mathfrak{p},\ z \in F_\mathfrak{p}^\times. \tag{2.17}$$

Here $u \in \Sigma_\mathfrak{p}$ acts on $V(\underline{n}, \underline{v})$ by identifying it as an element of $\Sigma_p = \prod_{\mathfrak{q}|p} \Sigma_\mathfrak{q}$ which is u at the factor at $\mathfrak{p}$, and identity at $\mathfrak{q} \neq \mathfrak{p}$ and $\gamma_p = \gamma_\mathfrak{p}\gamma_p^\mathfrak{p}$ where $\gamma_p^\mathfrak{p} = \prod_{\mathfrak{q}|p,\mathfrak{q}\neq\mathfrak{p}} \gamma_\mathfrak{q}$.

Formulas (2.8) and (2.10) for the action of the Hecke operators $T_\mathfrak{p}$ or $U_\mathfrak{p}$ can be applied verbatim to the components $(\phi_\mathfrak{p}^1, \cdots, \phi_\mathfrak{p}^{h_\mathfrak{p}})$, because the elements $x_{i,\mathfrak{p}}$ as above have trivial components at $\mathfrak{p}$. For instance, if $\mathfrak{p} \nmid \mathfrak{n}$, then the form $T_\mathfrak{p}\Phi$ corresponds to the $h_\mathfrak{p}$-tuple $(T_\mathfrak{p}\phi_\mathfrak{p}^1, \cdots, T_\mathfrak{p}\phi_\mathfrak{p}^{h_\mathfrak{p}})$, where

$$T_\mathfrak{p}\phi_\mathfrak{p}^i(g) = \sum_{a \in \mathbf{P}^1(k_\mathfrak{p})} \sigma_a(\mathfrak{p}) \cdot \phi_\mathfrak{p}^i\bigl(g\sigma_a(\mathfrak{p})\bigr).$$

Similarly if $\mathfrak{p}|\mathfrak{n}$, then

$$U_{\mathfrak{p}}\phi^i_{\mathfrak{p}}(g) = \sum_{a\in k_{\mathfrak{p}}} \widehat{\sigma}_a(\mathfrak{p}) \cdot \phi^i_{\mathfrak{p}}\big(g\widehat{\sigma}_a(\mathfrak{p})\big). \tag{2.18}$$

Suppose now that $\mathfrak{p}$ exactly divides $\mathfrak{m}$. We interpret this in terms of harmonic cocycles on the Bruhat–Tits tree.

Let $\mathcal{T}_{\mathfrak{p}}$ be the Bruhat–Tits tree of $\mathrm{PGL}_2(F_{\mathfrak{p}})$. Thus the vertex set of $\mathcal{T}_{\mathfrak{p}}$ is given by the set of homothety classes of lattices of $F_{\mathfrak{p}} \oplus F_{\mathfrak{p}}$. Denote by $\mathcal{V}(\mathcal{T}_{\mathfrak{p}})$ and $\mathcal{E}(\mathcal{T}_{\mathfrak{p}})$ the set of vertices and the set of oriented edges of $\mathcal{T}_{\mathfrak{p}}$ respectively. If $e \in \mathcal{E}(\mathcal{T}_{\mathfrak{p}})$, then we denote by $\bar{e}$ the opposite edge of e. The source and target vertices of e will be noted as $s(e)$ and $t(e)$. As usual $\mathcal{T}_{\mathfrak{p}}$ has the homogeneous (left) action of $\mathrm{PGL}_2(F_{\mathfrak{p}})$, and we can identify $\mathcal{V}(\mathcal{T}_{\mathfrak{p}}) = \mathrm{PGL}_2(F_{\mathfrak{p}})/\mathrm{GL}_2(\mathcal{O}_{F_{\mathfrak{p}}})$, and $\mathcal{E}(\mathcal{T}_{\mathfrak{p}}) = \mathrm{PGL}_2(F_{\mathfrak{p}})/I_{\mathfrak{p}}$.

Definition 2.6. A harmonic one-cocycle on $\mathcal{T}_{\mathfrak{p}}$ with values in $V = V(\underline{n}, \underline{v})$, is a function

$$c : \mathcal{E}(\mathcal{T}_{\mathfrak{p}}) \to V$$

such that the following two conditions hold: for any edge e:

$$c(\bar{e}) = -c(e) \tag{2.19}$$

and for all vertices v,

$$\sum_{s(e)=v} c(e) = 0. \tag{2.20}$$

The $\mathbf{C}_p$-vector space of V-valued harmonic one-cocycles will be noted as $C^1_{\mathrm{har}}(\mathcal{T}_{\mathfrak{p}}, V)$.

Denote by $\mathcal{O}_F^{(\mathfrak{p})}$ the ring of $\mathfrak{p}$-integers of F and by $(\mathcal{O}_F^{(\mathfrak{p})})^\times$ the group of $\mathfrak{p}$-units of F. We have $\widetilde{\Gamma}_i^{\mathfrak{p}} \cap F^\times = (\mathcal{O}_F^{(\mathfrak{p})})^\times$. Also note that if $\gamma \in \widetilde{\Gamma}_i^{\mathfrak{p}}$, then $\mathrm{Nrd}_{B/F}\,\gamma \in (\mathcal{O}_F^{(\mathfrak{p})})^\times \cap F_+^\times$. We have the action of $B^\times$ on V via the map $B^\times \to \prod_{\mathfrak{q}|p} B_{\mathfrak{q}}^\times$, but we want to twist this so that the resulting action factors through the quotient of $B^\times$ by $(\mathcal{O}_F^{(\mathfrak{p})})^\times$.

Define the following action $\star_{\mathfrak{p}}$ of $B^\times$ on V: for $\gamma \in B^\times$, and $v \in V$,

$$\gamma \star_{\mathfrak{p}} v := |\,\mathrm{Nrd}_{B/F}\,\gamma|_{\mathfrak{p}}^{m/2}\gamma_p \cdot v.$$

Then the action the $\star_{\mathfrak{p}}$ factors through $B^\times/(\mathcal{O}_F^{(\mathfrak{p})})^\times$. For $i = 1, \cdots, h_{\mathfrak{p}}$, put $\Gamma_i^{\mathfrak{p}} = \widetilde{\Gamma}_i^{\mathfrak{p}}/(\mathcal{O}_F^{(\mathfrak{p})})^\times$.

The space $C^1_{\mathrm{har}}(\mathcal{T}_{\mathfrak{p}}, V)$ supports the following action of $B^\times/(\mathcal{O}_F^{(\mathfrak{p})})^\times$: for $\gamma \in B^\times$, and $c \in C^1_{\mathrm{har}}(\mathcal{T}_{\mathfrak{p}}, V)$, the cocycle $\gamma \star_{\mathfrak{p}} c$ is defined by

$$(\gamma \star_{\mathfrak{p}} c)(e) = \gamma \star_{\mathfrak{p}} \big(c\big(\gamma_{\mathfrak{p}}^{-1}e\big)\big).$$

Now let $\varPhi \in S^B_{\underline{n},\underline{v}}(\mathfrak{m}\mathfrak{n}^+, \mathfrak{n}^-)$ which corresponds to an $h_\mathfrak{p}$-tuple: $(\phi^1_\mathfrak{p}, \cdots, \phi^{h_\mathfrak{p}}_\mathfrak{p})$ as above. Assume that $\mathfrak{p}$ exactly divides $\mathfrak{m}$. For each $i = 1, \cdots, h_\mathfrak{p}$, define a function $c_{\phi^i_\mathfrak{p}}$ on $\mathcal{E}(\mathcal{T}_\mathfrak{p})$ as follows.

Let $e = (s, t) \in \mathcal{E}(\mathcal{T}_\mathfrak{p})$ (going from s to t). Represent s and t by lattices L_s and L_t such that L_s contains L_t with index $\mathcal{N}\mathfrak{p}$. Let $g_e \in \mathrm{GL}_2(F_\mathfrak{p})$ such that $g_e(\mathcal{O}_{F_\mathfrak{p}} \oplus \mathcal{O}_{F_\mathfrak{p}}) = L_s$ and $g_e(\mathcal{O}_{F_\mathfrak{p}} \oplus \mathfrak{p}\mathcal{O}_{F_\mathfrak{p}}) = L_t$. Then define

$$c_{\phi^i_\mathfrak{p}}(e) = |\det g_e|^{m/2}_\mathfrak{p} g_e \cdot \phi^i_\mathfrak{p}(g_e). \tag{2.21}$$

By (2.6) and (2.17) this is well-defined independent of the choice of L_s, L_t, and independent of the choice of g_e. The following property also follows from (2.17):

$$c_{\phi^i_\mathfrak{p}}(\gamma_\mathfrak{p} e) = \gamma \star_\mathfrak{p} c_{\phi^i_\mathfrak{p}}(e) \quad \text{for } \gamma \in \varGamma^\mathfrak{p}_i. \tag{2.22}$$

Denote by $c_{\varPhi,\mathfrak{p}}$ the vector of functions $\{c_{\phi^i_\mathfrak{p}}\}^{h_\mathfrak{p}}_{i=1}$.

The $U_\mathfrak{p}$ operator has the following combinatorial description:

$$c_{U_\mathfrak{p}\phi^i_\mathfrak{p}}(e) = \mathcal{N}\mathfrak{p}^{m/2} \sum_{s(e')=t(e)} c_{\phi^i_\mathfrak{p}}(e'). \tag{2.23}$$

Similarly for the Atkin–Lehner operator $W_\mathfrak{p}$:

$$c_{W_\mathfrak{p}\phi^i_\mathfrak{p}}(e) = \mathcal{N}\mathfrak{p}^{m/2} c_{\phi^i_\mathfrak{p}}(\bar{e}). \tag{2.24}$$

Proposition 2.7. *Suppose that the form $\varPhi \in S^B_{\underline{n},\underline{v}}(\mathfrak{m}\mathfrak{n}^+, \mathfrak{n}^-)$ is new at $\mathfrak{p}$, and satisfies the condition*

$$U_\mathfrak{p}\varPhi = \mathcal{N}\mathfrak{p}^{m/2}\varPhi. \tag{2.25}$$

Then the functions $c_{\phi^i_\mathfrak{p}}$ are in $C^1_{\mathrm{har}}(\mathcal{T}_\mathfrak{p}, V)$, and are invariant under $\varGamma^\mathfrak{p}_i$ (with respect to the action $\star_\mathfrak{p}$).

Proof. Condition (2.20) follows from (2.14), (2.23), and (2.24), together with the assumption that $\varPhi$ is new at $\mathfrak{p}$. With (2.20), condition (2.19) then follows from (2.23) and condition (2.25). The invariance under $\varGamma^\mathfrak{p}_i$ is a restatement of (2.22). □

Remark 2.8. It can be readily checked that if $\varPhi' \in S_{\underline{n},\underline{v}'}(\mathfrak{m}\mathfrak{n}^+, \mathfrak{n}^-)$ is the form obtained from $\varPhi$ by the twisting operation as described in Remark 2.4, then $\varPhi$ and $\varPhi'$ define the same harmonic cocycle (with the identification of the underlying vector space of $V(\underline{n}, \underline{v})$ and $V(\underline{n}, \underline{v}')$).

In general let $\varGamma$ be a subgroup of $B^\times/(\mathcal{O}^{(\mathfrak{p})}_F)^\times$. We now define Schneider's map from the space of V-valued $\varGamma$-invariant cocycles on $\mathcal{T}_\mathfrak{p}$ to $H^1(\varGamma, V)$. Choose a vertex $v \in \mathcal{V}(\mathcal{T}_\mathfrak{p})$.

For $c \in C^1_{\mathrm{har}}(\mathcal{T}_{\mathfrak{p}}, V)^{\Gamma}$, define κ_c to be the following function on Γ with values in V: for $\gamma \in \Gamma$,

$$\kappa_c(\gamma) = \sum_{e: v \to \gamma_{\mathfrak{p}} v} c(e), \tag{2.26}$$

where the sum runs over the edges in the geodesic joining v and γv.

From the Γ-invariance of c, it follows that κ_c is a one-cocycle. Furthermore, the class of κ_c in $H^1(\Gamma, V)$ is independent of the choice of v.

Now apply this construction to $\Gamma = \Gamma_i^{\mathfrak{p}}$. Denote by:

$$\begin{aligned} \kappa_{i,\mathfrak{p}}^{\mathrm{sch}} : C^1_{\mathrm{har}}(\mathcal{T}_{\mathfrak{p}}, V)^{\Gamma_i^{\mathfrak{p}}} &\to H^1\big(\Gamma_i^{\mathfrak{p}}, V\big) \\ c &\mapsto \text{class of } \kappa_c \end{aligned}$$

the map defined as above. We also denote by

$$\kappa_{\mathfrak{p}}^{\mathrm{sch}} : \bigoplus_{i=1}^{h_{\mathfrak{p}}} C^1_{\mathrm{har}}(\mathcal{T}_{\mathfrak{p}}, V)^{\Gamma_i^{\mathfrak{p}}} \to \bigoplus_{i=1}^{h_{\mathfrak{p}}} H^1\big(\Gamma_i^{\mathfrak{p}}, V\big)$$

the direct sum of the maps $\kappa_{i,\mathfrak{p}}^{\mathrm{sch}}$. We denote by $\kappa_{i,\mathfrak{p}}^{\mathrm{sch}}(\phi_{\mathfrak{p}}^i) \in H^1(\Gamma_i^{\mathfrak{p}}, V)$ the image of $c_{\phi_{\mathfrak{p}}^i}$, under $\kappa_{i,\mathfrak{p}}^{\mathrm{sch}}$. Similarly denote by $\kappa_{\mathfrak{p}}^{\mathrm{sch}}(\Phi)$ the image of $c_{\Phi,\mathfrak{p}}$ under $\kappa_{\mathfrak{p}}^{\mathrm{sch}}$. Note that by construction we have an identification of $\bigoplus_{i=1}^{h_{\mathfrak{p}}} C^1_{\mathrm{har}}(\mathcal{T}_{\mathfrak{p}}, V)^{\Gamma_i^{\mathfrak{p}}}$ with the eigenspace of $S^B_{\underline{n},\underline{v}}(\mathfrak{m}\mathfrak{n}^+, \mathfrak{n}^-)^{\mathfrak{p}-new}$ with $U_{\mathfrak{p}}$-eigenvalue equal to $\mathcal{N}\mathfrak{p}^{m/2}$.

To conclude this section we state:

Proposition 2.9. *The maps $\kappa_{i,\mathfrak{p}}^{\mathrm{sch}}$ are isomorphisms. Hence the map $\kappa_{\mathfrak{p}}^{\mathrm{sch}}$ is an isomorphism.*

Without interrupting the main reasoning of the paper we refer the reader to Appendix A for the proof.

2.5. Action of Hecke operators and Jacquet–Langlands–Shimizu correspondence

When we use the strong approximation theorem with respect to the prime $\mathfrak{p}$ (Eqs. (2.15)–(2.17)) for the description of automorphic forms, the action of the Hecke operators $T_{\mathfrak{l}}$ or $U_{\mathfrak{l}}$ for $\mathfrak{l} \neq \mathfrak{p}$ becomes more complicated as compared to the adelic description (when $\mathfrak{l} = \mathfrak{p}$ it is as in (2.18)). We describe this in the first part of this section. The reader familiar with the formalism of [14, Section 2], will notice the similarities.

First we observe the following: let $y \in \widehat{B}^{\times}$ whose component at $\mathfrak{p}$ is trivial. Then by (2.15), for any $j = 1, \cdots, h_{\mathfrak{p}}$, there is a unique $i = i(j) \in \{1, \cdots, h_{\mathfrak{p}}\}$ corresponding to j, and $\alpha \in B^{\times}$, $b \in \mathrm{GL}_2(F_{\mathfrak{p}})$, and $u \in \Sigma^{\mathfrak{p}}$ (here $\Sigma^{\mathfrak{p}} = \prod_{\mathfrak{l} \neq \mathfrak{p}} \Sigma_{\mathfrak{l}}$), such that

$$x_{j,\mathfrak{p}} y = \alpha x_{i,\mathfrak{p}} b u. \tag{2.27}$$

Since the $x_{i,\mathfrak{p}}$'s have trivial component at $\mathfrak{p}$, (2.27) is equivalent to:

$$\begin{aligned} x_{j,\mathfrak{p}}y &= \alpha^{\mathfrak{p}} x_{i,\mathfrak{p}} u \\ \alpha_{\mathfrak{p}} &= b^{-1}. \end{aligned} \tag{2.28}$$

Now let $\mathfrak{l} \neq \mathfrak{p}$. Put $y_{\mathfrak{l}} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \pi_{\mathfrak{l}} \end{smallmatrix}\right)$ if $\mathfrak{l} \nmid \mathfrak{n}^-$, and $y_{\mathfrak{l}} = \omega_{\mathfrak{l}}$ if $\mathfrak{l} | \mathfrak{n}^-$ (with $\omega_{\mathfrak{l}}$ a uniformizer of $R_{\mathfrak{l}}$). In either case we regard $y_{\mathfrak{l}}$ as an element of $\widehat{B}^\times$ that has trivial components at places outside $\mathfrak{l}$.

Take $y = y_{\mathfrak{l}}$ in (2.27), (2.28). Then

$$\begin{aligned} y_{\mathfrak{l}} &= (x_{j,\mathfrak{p}})_{\mathfrak{l}}^{-1} \alpha_{\mathfrak{l}} (x_{i,\mathfrak{p}})_{\mathfrak{l}} u_{\mathfrak{l}}, \\ 1 &= (x_{j,\mathfrak{p}})_{\mathfrak{l}'}^{-1} \alpha_{\mathfrak{l}'} (x_{i,\mathfrak{p}})_{\mathfrak{l}'} u_{\mathfrak{l}'} \quad \text{if } \mathfrak{l}' \neq \mathfrak{l}, \mathfrak{p} \end{aligned}$$

which implies

$$\Sigma_{\mathfrak{l}} y_{\mathfrak{l}} \Sigma_{\mathfrak{l}} = \Sigma_{\mathfrak{l}} (x_{j,\mathfrak{p}})_{\mathfrak{l}}^{-1} \alpha_{\mathfrak{l}} (x_{i,\mathfrak{p}})_{\mathfrak{l}} \Sigma_{\mathfrak{l}}. \tag{2.29}$$

Proposition 2.10. *Suppose that we have a double coset decomposition*

$$\widetilde{\Gamma}_j^{\mathfrak{p}} \alpha \widetilde{\Gamma}_i^{\mathfrak{p}} = \bigsqcup_r \alpha_r \widetilde{\Gamma}_i^{\mathfrak{p}}. \tag{2.30}$$

Then we have a corresponding double coset decomposition

$$\Sigma_{\mathfrak{l}} y_{\mathfrak{l}} \Sigma_{\mathfrak{l}} = \bigsqcup_r (x_{j,\mathfrak{p}})_{\mathfrak{l}}^{-1} (\alpha_r)_{\mathfrak{l}} (x_{i,\mathfrak{p}})_{\mathfrak{l}} \Sigma_{\mathfrak{l}}. \tag{2.31}$$

Proof. This can be proved using the strong approximation theorem, as in [14, (2.8a)–(2.8b)]. □

Suppose that $\mathfrak{l} \nmid \mathfrak{n}$. Then from (2.31) we see that the elements $(x_{j,\mathfrak{p}})_{\mathfrak{l}}^{-1} (\alpha_r)_{\mathfrak{l}} (x_{i,\mathfrak{p}})_{\mathfrak{l}}$ can be taken to be the matrices $\sigma_r(\mathfrak{l})$ used for defining $T_{\mathfrak{l}}$ as in (2.7). Thus if we put $\sigma_r(\mathfrak{l}) = (x_{j,\mathfrak{p}})_{\mathfrak{l}}^{-1} (\alpha_r)_{\mathfrak{l}} (x_{i,\mathfrak{p}})_{\mathfrak{l}}$, then it is easy to check that we can write $x_{j,\mathfrak{p}} \sigma_r(\mathfrak{l})$ in the form (2.27). More precisely,

$$x_{j,\mathfrak{p}} \sigma_r(\mathfrak{l}) = \alpha_r x_{i,\mathfrak{p}} b_r u_r$$

for some $b_r \in \mathrm{GL}_2(F_{\mathfrak{p}})$, $u_r \in \Sigma^{\mathfrak{p}}$. Similar discussions hold when $\mathfrak{l} | \mathfrak{n}$.

Proposition 2.11. *Let $\Phi \in S^B_{\underline{n},\underline{v}}(\mathfrak{m}\mathfrak{n}^+, \mathfrak{n}^-)$ correspond to the $h_{\mathfrak{p}}$-tuple $(\phi^1, \cdots, \phi^{h_{\mathfrak{p}}})$. For $\mathfrak{l} \neq \mathfrak{p}$, let $\widetilde{\Phi} = T_{\mathfrak{l}} \Phi$ or $U_{\mathfrak{l}} \Phi$ depending on whether $\mathfrak{l} \nmid \mathfrak{n}$ or $\mathfrak{l} | \mathfrak{n}$, and denote by $(\widetilde{\phi}^1, \cdots, \widetilde{\phi}^{h_{\mathfrak{p}}})$ the corresponding $h_{\mathfrak{p}}$-tuple. Then given an index j, we have in the above notations:*

$$\widetilde{\phi}^j(g) = \sum_r (\alpha_r)_p^{\mathfrak{p}} \cdot \phi^i\big((\alpha_r)_{\mathfrak{p}}^{-1} g\big) \quad \textit{for } g \in \mathrm{GL}_2(F_{\mathfrak{p}}). \tag{2.32}$$

(So in particular the action of $T_{\mathfrak{l}}$ or $U_{\mathfrak{l}}$ "permutes the components" of Φ.)

Proof. This is a direct computation. First notice that if y satisfies (2.27)–(2.28) as above, we have for any $g \in \mathrm{GL}_2(F_\mathfrak{p})$:

$$\begin{aligned}\Phi(x_{j,\mathfrak{p}}yg) &= \Phi\big(\alpha^\mathfrak{p} x_{i,\mathfrak{p}}ug\big)\\ &= \Phi\big(\alpha^{-1}\alpha^\mathfrak{p} x_{i,\mathfrak{p}}gu\big)\\ &= u_p^{-1}\cdot\Phi\big(\alpha_\mathfrak{p}^{-1}x_{i,\mathfrak{p}}g\big)\\ &= \big(y_p^{-1}\alpha_p^\mathfrak{p}\big)\cdot\Phi\big(x_{i,\mathfrak{p}}\alpha_\mathfrak{p}^{-1}g\big).\end{aligned}$$

(For the last equality note that y and u have trivial component at $\mathfrak{p}$, and that x_j, x_i have trivial components at all primes above p.)

Consider the case $\mathfrak{l}\nmid\mathfrak{n}$ for $T_\mathfrak{l}$. The case where $\mathfrak{l}|\mathfrak{n}$ for $U_\mathfrak{l}$ is similar. We have:

$$\begin{aligned}\widetilde{\phi}^j(g) &= \widetilde{\Phi}(x_{j,\mathfrak{p}}g)\\ &= (T_\mathfrak{l}\Phi)(x_{j,\mathfrak{p}}g)\\ &= \sum_r\big(\sigma_r(\mathfrak{l})\big)_p\cdot\Phi\big(x_{j,\mathfrak{p}}g\sigma_r(\mathfrak{l})\big)\\ &= \sum_r\big(\sigma_r(\mathfrak{l})\big)_p\cdot\Phi\big(x_{j,\mathfrak{p}}\sigma_r(\mathfrak{l})g\big).\end{aligned}$$

By the above computations applied to $y=\sigma_r(\mathfrak{l})$, we have

$$\big(\sigma_r(\mathfrak{l})\big)_p\cdot\Phi\big(x_{j,\mathfrak{p}}\sigma_r(\mathfrak{l})g\big) = (\alpha_r)_p^\mathfrak{p}\cdot\Phi\big(x_{i,\mathfrak{p}}\alpha_\mathfrak{p}^{-1}g\big) = \alpha_p^\mathfrak{p}\cdot\phi^i\big(\alpha_\mathfrak{p}^{-1}g\big)$$

and the result follows. $\square$

We can similarly define action of the Hecke operators $T_\mathfrak{l}$ or $U_\mathfrak{l}$ ($\mathfrak{l}\neq\mathfrak{p}$) on the spaces $\bigoplus_{i=1}^{h_\mathfrak{p}} C^1_{\mathrm{har}}(\mathcal{T}_\mathfrak{p},V)^{\Gamma_i^\mathfrak{p}}$ and $\bigoplus_{i=1}^{h_\mathfrak{p}} H^1(\Gamma_i^\mathfrak{p},V)$. For instance, if $\underline{c}=(c_1,\cdots,c_{h_\mathfrak{p}})\in\bigoplus_{i=1}^{h_\mathfrak{p}} C^1_{\mathrm{har}}(\mathcal{T}_\mathfrak{p},V)^{\Gamma_i^\mathfrak{p}}$, define $\widetilde{\underline{c}}=T_\mathfrak{l}\underline{c}=(\widetilde{c}_1,\cdots,\widetilde{c}_{h_\mathfrak{p}})$, where (in the above notations):

$$\widetilde{c}_j(e)=\sum_r\alpha_r\star_\mathfrak{p} c_i\big((\alpha_r)_\mathfrak{p}^{-1}e\big). \tag{2.33}$$

An immediate computation shows that if $\underline{c}$ is the $h_\mathfrak{p}$-tuple of harmonic cocycles associated to Φ (as in (2.21)), then $T_\mathfrak{l}\underline{c}$ is the $h_\mathfrak{p}$-tuple associated to $T_\mathfrak{l}\Phi$.

Similarly, if $\underline{\kappa}=(\kappa_1,\cdots,\kappa_{h_\mathfrak{p}})\in\bigoplus_{i=1}^{h_\mathfrak{p}} H^1(\Gamma_i^\mathfrak{p},V)$, define $\widetilde{\underline{\kappa}}=T_\mathfrak{l}\underline{\kappa}=(\widetilde{\kappa}_1,\cdots,\widetilde{\kappa}_{h_\mathfrak{p}})$ as follows. In the notations above, for the index j and $\gamma\in\widetilde{\Gamma}_j$ (and we continue to denote by γ its image in $\Gamma_j^\mathfrak{p}$), then:

$$\widetilde{\kappa}_j(\gamma)=\sum_r\alpha_r\star_\mathfrak{p}\kappa_i\big(\alpha_r^{-1}\gamma\alpha_{r'}\big) \tag{2.34}$$

here r' is the unique index in (2.30) such that $\alpha_r^{-1}\gamma\alpha_{r'} \in \widetilde{\Gamma}_i$ (thus the map $r \mapsto r'$ is a permutation). Note that (2.34) is well-defined independent of the choice of the cocycles representing the κ_i's.

Again we remark that the action of $T_{\mathfrak{l}}$ and $U_{\mathfrak{l}}$ permutes the components of $\bigoplus_{i=1}^{h_{\mathfrak{p}}} C^1_{\mathrm{har}}(\mathcal{T}_{\mathfrak{p}}, V)^{\Gamma_i^{\mathfrak{p}}}$ and that of $\bigoplus_{i=1}^{h_{\mathfrak{p}}} H^1(\Gamma_i^{\mathfrak{p}}, V)$.

Proposition 2.12. *The isomorphism* $\kappa_{\mathfrak{p}}^{\mathrm{sch}} : \bigoplus_{i=1}^{h_{\mathfrak{p}}} C^1_{\mathrm{har}}(\mathcal{T}_{\mathfrak{p}}, V)^{\Gamma_i^{\mathfrak{p}}} \to \bigoplus_{i=1}^{h_{\mathfrak{p}}} H^1(\Gamma_i^{\mathfrak{p}}, V)$ *(cf. Proposition 2.9) commutes the action of* $T_{\mathfrak{l}}$ *and* $U_{\mathfrak{l}}$ *for all* $\mathfrak{l} \neq \mathfrak{p}$.

Proof. This is a direct computation. Let $\underline{c} \in \bigoplus_{i=1}^{h_{\mathfrak{p}}} C^1_{\mathrm{har}}(\mathcal{T}_{\mathfrak{p}}, V)^{\Gamma_i^{\mathfrak{p}}}$, denote by $\underline{\kappa} \in \bigoplus_{i=1}^{h_{\mathfrak{p}}} H^1(\Gamma_i^{\mathfrak{p}}, V)$ the image of $\underline{c}$ under $\kappa_{\mathfrak{p}}^{\mathrm{sch}}$, any $\underline{\kappa}'$ the image of $T_{\mathfrak{l}}\underline{c}$ (or $U_{\mathfrak{l}}\underline{c}$ if $\mathfrak{l}|\mathfrak{n}$) under $\kappa_{\mathfrak{p}}^{\mathrm{sch}}$. We maintain the notation of (2.33). Write $\gamma' := \alpha_r^{-1}\gamma\alpha_{r'} \in \widetilde{\Gamma}_i$. Fix $v_0 \in \mathcal{V}(\mathcal{T}_{\mathfrak{p}})$; at the level of cocycles we compute (to avoid notational difficulty we omit the subscript $\mathfrak{p}$ in the computations):

$$\begin{aligned}
\kappa'_j(\gamma) &= \sum_{e: v_0 \to \gamma v_0} \widetilde{c}_j(e) \\
&= \sum_{e: v_0 \to \gamma v_0} \sum_r \alpha_r \star_{\mathfrak{p}} c_i\big(\alpha_r^{-1} e\big) \\
&= \sum_r \sum_{e: \alpha_r^{-1} v_0 \to \alpha_r^{-1}\gamma v_0} \alpha_r \star_{\mathfrak{p}} c_i(e) \\
&= \sum_r \sum_{e: \alpha_r^{-1} v_0 \to \gamma'\alpha_{r'}^{-1} v_0} \alpha_r \star_{\mathfrak{p}} c_i(e) \\
&= \sum_r \sum_{e: \alpha_r^{-1} v_0 \to v_0} \alpha_r \star_{\mathfrak{p}} c_i(e) + \sum_r \sum_{e: v_0 \to \gamma' v_0} \alpha_r \star_{\mathfrak{p}} c_i(e) \\
&\quad + \sum_r \sum_{e: \gamma' v_0 \to \gamma'\alpha_{r'}^{-1} v_0} \alpha_r \star_{\mathfrak{p}} c_i(e).
\end{aligned}$$

Now the term

$$\begin{aligned}
\sum_r \sum_{e: \gamma' v_0 \to \gamma'\alpha_{r'}^{-1} v_0} \alpha_r \star_{\mathfrak{p}} c_i(e) &= \sum_r \sum_{e: v_0 \to \alpha_{r'}^{-1} v_0} \alpha_r \star_{\mathfrak{p}} c_i\big(\gamma' e\big) \\
&= \sum_r \sum_{e: v_0 \to \alpha_{r'}^{-1} v_0} \big(\alpha_r \gamma'\big) \star_{\mathfrak{p}} c_i(e) \\
&= \sum_r \sum_{e: v_0 \to \alpha_{r'}^{-1} v_0} (\gamma\alpha_{r'}) \star_{\mathfrak{p}} c_i(e) \\
&= \gamma \star_{\mathfrak{p}} \Bigg(\sum_{r'} \sum_{e: v_0 \to \alpha_{r'}^{-1} v_0} \alpha_{r'} \star_{\mathfrak{p}} c_i(e) \Bigg).
\end{aligned}$$

Hence if we put

$$X := \sum_r \sum_{e: v_0 \to \alpha_r^{-1} v_0} \alpha_r \star_{\mathfrak{p}} c_i(e) \in V$$

(which is independent of γ), then

$$\kappa_j'(\gamma) = \sum_r \alpha_r \star_{\mathfrak{p}} \kappa_i(\gamma') + \gamma \star_{\mathfrak{p}} X - X.$$

Passing to cohomology we obtain $\underline{\kappa}' = T_{\mathfrak{l}} \underline{\kappa}$ (or $U_{\mathfrak{l}} \underline{\kappa}$ if $\mathfrak{l} | \mathfrak{n}$) as required. □

In the remainder of this subsection, we recall the relation between the quaternionic forms of Section 2.1 and Hilbert modular forms via the Jacquet–Langlands correspondence. First recall some definitions regarding Hilbert modular forms. For more details, see [7, Chapter 2].

We will generally denote a place corresponding to an embedding of F into $\mathbf{C}$ as ν. Recall that we have fixed an embedding of $\overline{\mathbf{Q}}$ into $\overline{\mathbf{Q}}_p$ and $\mathbf{C}$, which allows us to identify I also as the set of embeddings of F into $\mathbf{C}$ (which necessarily has image in $\mathbf{R}$).

Notation 2.13. Put $F_\infty = F \otimes_{\mathbf{Q}} \mathbf{R}$, the archimedean component of $\mathbf{A}_F$. For $x \in \mathbf{A}_F$, denote by x_∞ its archimedean component, and we denote by $x_\infty \gg 0$ the condition of total positivity, i.e. all components at the infinite places are positive.

For $x_\infty = (x_\nu)_{\nu \in I} \in F_\infty$, put

$$e_F(x_\infty) = \exp\left(2\pi i \sum_{\nu \in I} x_\nu\right)$$

and

$$e_F(\mathbf{i} x_\infty) = \exp\left(-2\pi \sum_{\nu \in I} x_\nu\right).$$

Let ψ_F be the standard unitary additive character of $\mathbf{A}_F/F$ such that $\psi_F(x_\infty) = e_F(x_\infty)$ for $x_\infty \in F_\infty$.

For any $y \in \mathbf{A}_F^\times$, denote by $y\mathcal{O}_F$ the fractional ideal associated to y.

Put

$$K(\mathfrak{n}) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathcal{O}}_F) \,\middle|\, \gamma \in \mathfrak{n}\widehat{\mathcal{O}}_F \right\}.$$

This will be the level group in the Hilbert modular case.

Definition 2.14. Let $\mathfrak{n}$ be an ideal of $\mathcal{O}_F$, and $\underline{k}, \underline{v} \in \mathbf{Z}[I]$ with $k_\nu \geq 2$, satisfying $\underline{k} + 2\underline{v} = (m+2)\underline{t}$ for some integer m. By a cuspidal Hilbert modular form of weight $(\underline{k}, \underline{v})$ and level $\mathfrak{n}$ (of trivial nebentype), we mean a function $\mathbf{f} : \mathrm{GL}_2(\mathbf{A}_F) \to \mathbf{C}$, satisfying the following conditions:

(1) $\mathbf{f}$ satisfies the following transformation properties:

$$\mathbf{f}(zsg) = |z|_{\mathbf{A}_F}^{-m}\mathbf{f}(g) \quad \text{for all } g \in \mathrm{GL}_2(\mathbf{A}_F),\ s \in \mathrm{GL}_2(F),\ z \in \mathbf{A}_F^{\times},$$
$$\mathbf{f}\big(gr(\underline{\theta})\big) = \mathbf{f}(g)\prod_{\nu \in I} e^{ik_\nu\theta_\nu}$$

where $r(\underline{\theta}) = (r_\nu(\theta_\nu))_{\nu\in I} \in \prod_{\nu\in I}\mathrm{SO}(2)$, with

$$r_\nu(\theta_\nu) = \begin{pmatrix} \cos(\theta_\nu) & \sin(\theta_\nu) \\ -\sin(\theta_\nu) & \cos(\theta_\nu) \end{pmatrix},$$
$$\mathbf{f}(g\kappa) = \mathbf{f}(g) \quad \text{for all } g \in \mathrm{GL}_2(\mathbf{A}_F),\ \kappa \in K(\mathfrak{n}).$$

(2) At each archimdean place ν, the form $\mathbf{f}$ generates the discrete series representation of $\mathrm{GL}_2(F_\nu)$ of weight k_ν.
(3) $\mathbf{f}$ satisfies the cuspidal condition

$$\int\limits_{N(F)\backslash N(\mathbf{A}_F)} \mathbf{f}(ng)\,dn = 0 \quad \text{for all } g \in \mathrm{GL}_2(\mathbf{A}_F),$$

where $N \subset \mathrm{GL}_2$ is the subgroup of upper triangular unipotent matrices.

The complex vector space of cuspidal Hilbert modular forms of weight $(\underline{k}, \underline{v})$, and level $\mathfrak{n}$, is denoted as $S_{\underline{k},\underline{v}}(\mathfrak{n}, \mathbf{C})$.

Remark 2.15. In Definition 2.14, for condition (1) to be consistent k_ν has to be even (hence also m) for all $\nu \in I$. If $\underline{v} = \underline{0}$, then $\mathbf{f}$ is said to have parallel weight $\underline{k}$. We note here that the definition employed here is slightly different from Definition 2.1 of [11], most notably concerning the central character. The definition in [11] was convenient for parallel weight forms, but for non-parallel weights situation the present definition is more convenient.

Remark 2.16. If $\mathbf{f} \in S_{\underline{k},\underline{v}}(\mathfrak{n}, \mathbf{C})$, then for an integer r, the form $\mathbf{f}'$ defined by

$$\mathbf{f}'(g) = |\det g|_{\mathbf{A}_F}^{-r}\mathbf{f}(g) \quad \text{for } g \in \mathrm{GL}_2(\mathbf{A}_F) \tag{2.35}$$

lies in $S_{\underline{k},\underline{v}'}(\mathfrak{n}, \mathbf{C})$, where $\underline{v}' = \underline{v} + r\underline{t}$.

Given a form $\mathbf{f} \in S_{\underline{k},\underline{v}}(\mathfrak{n}, \mathbf{C})$, we have the adelic Fourier expansion: let d be an idele of F whose associated ideal is the different of F. Then for all $y \in \mathbf{A}_F^\times$, whose archimedean component y_∞ is totally positive, and $x \in \mathbf{A}_F$,

$$\mathbf{f}\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = |y|_{\mathbf{A}_F} \sum_{0 \ll \xi \in F} (\xi_\infty y_\infty)^{-\{\underline{v}\}} C(\xi y d \mathcal{O}_F, \mathbf{f}) \psi_F(\xi x) e_F(\mathbf{i}\xi_\infty y_\infty). \tag{2.36}$$

(Here for $y \in \mathbf{R}^I$, with I identifying as the set of embeddings of F into $\mathbf{R}$, the notation $y^{\{\underline{v}\}}$ stands for $\prod_{\nu \in I} y_\nu^{v_\nu}$.)

The coefficients $C(\mathfrak{b}, \mathbf{f})$ range over all the integral ideals $\mathfrak{b}$ of $\mathcal{O}_F$ (and are understood to be zero if $\mathfrak{b}$ is not integral), and are called the normalized Fourier coefficients of $\mathbf{f}$. The form $\mathbf{f}$ is called normalized if $C(\mathcal{O}_F, \mathbf{f}) = 1$.

Remark 2.17. In the situation of Remark 2.16, we have for all ideal $\mathfrak{b}$ of $\mathcal{O}_F$,

$$C\left(\mathfrak{b}, \mathbf{f}'\right) = \mathcal{N}\mathfrak{b}^r C(\mathfrak{b}, \mathbf{f}). \tag{2.37}$$

Let $\widetilde{F}$ be the composite of the image of F under all elements of $\mathrm{Hom}(F, \overline{\mathbf{Q}})$. For any subfield E of $\mathbf{C}$ that contains $\widetilde{F}$, define

$$S_{\underline{k},\underline{v}}(\mathfrak{n}, E) = \left\{\mathbf{f} \in S_{\underline{k},\underline{v}}(\mathfrak{n}, \mathbf{C}) \;\middle|\; C(\mathfrak{b}, \mathbf{f}) \in E \text{ for all } \mathfrak{b}\right\}.$$

Then $S_{\underline{k},\underline{v}}(\mathfrak{n}, E)$ is an E-vector space, and it is a theorem of Shimura [14] that

$$S_{\underline{k},\underline{v}}(\mathfrak{n}, \mathbf{C}) = S_{\underline{k},\underline{v}}(\mathfrak{n}, E) \otimes_E \mathbf{C}. \tag{2.38}$$

Hence one can define $S_{\underline{k},\underline{v}}(\mathfrak{n}, E)$, for any field E that contains $\widetilde{F}$ (in particular for $E = \overline{\mathbf{Q}}, \overline{\mathbf{Q}}_p, \mathbf{C}_p$), by

$$S_{\underline{k},\underline{v}}(\mathfrak{n}, E) = S_{\underline{k},\underline{v}}(\mathfrak{n}, \widetilde{F}) \otimes_{\widetilde{F}} E.$$

For $\mathfrak{l} \nmid \mathfrak{n}$, one has Hecke operators $T_\mathfrak{l}$ acting on $S_{\underline{k},\underline{v}}(\mathfrak{n}, \mathbf{C})$, such that on the normalized Fourier coefficients:

$$C(\mathfrak{b}, T_\mathfrak{l}\mathbf{f}) = C(\mathfrak{b}\mathfrak{l}, \mathbf{f}) + \mathcal{N}\mathfrak{l}^{m+1} C\left(\frac{\mathfrak{b}}{\mathfrak{l}}, \mathbf{f}\right).$$

If $\mathfrak{l} | \mathfrak{n}$, then one has the operators $U_\mathfrak{l}$, for which

$$C(\mathfrak{b}, U_\mathfrak{l}\mathbf{f}) = C(\mathfrak{b}\mathfrak{l}, \mathbf{f}).$$

There is also an operator

$$V_\mathfrak{l} : S_{\underline{k},\underline{v}}\left(\frac{\mathfrak{n}}{\mathfrak{l}}, \mathbf{C}\right) \to S_{\underline{k},\underline{v}}(\mathfrak{n}, \mathbf{C})$$

such that

$$C(\mathfrak{b}, V_{\mathfrak{l}}\mathbf{f}) = C\left(\frac{\mathfrak{b}}{\mathfrak{l}}, \mathbf{f}\right)$$

(here $C(\frac{\mathfrak{b}}{\mathfrak{l}}, \mathbf{f})$ is understood to be zero when $\mathfrak{l} \nmid \mathfrak{b}$). A form in $S_{\underline{k},\underline{v}}(\mathfrak{n}, \mathbf{C})$ is called new at $\mathfrak{l}$ if it does not lie in the span of the image of $S_{\underline{k},\underline{v}}(\frac{\mathfrak{n}}{\mathfrak{l}}, \mathbf{C})$ under the natural inclusion and under $V_{\mathfrak{l}}$.

If $\mathbf{f}$ is normalized, and is an eigenvector for $T_{\mathfrak{l}}$, then the eigenvalue is $C(\mathfrak{l}, \mathbf{f})$ (similarly for $U_{\mathfrak{l}}$). A Hilbert modular form $\mathbf{f} \in S_{\underline{k},\underline{v}}(\mathfrak{n}, \mathbf{C})$ is called an eigenform if it is normalized and is an eigenvector for all the Hecke operators, in which case we have $\mathbf{f} \in S_{\underline{k},\underline{v}}(\mathfrak{n}, \overline{\mathbf{Q}})$. It is called a newform if it is new at all primes dividing the level $\mathfrak{n}$, in which case $\mathfrak{n}$ is called the conductor of $\mathbf{f}$.

Suppose that $\mathbf{f} \in S_{\underline{k},\underline{v}}(\mathfrak{n}, \mathbf{C})$, with $\underline{k} + 2\underline{v} = (m + 2)\underline{t}$ as above, is a normalized eigenform. If χ is a finite order Hecke character of F, define the complex L-function of $\mathbf{f}$ and χ, for $Re(s) \gg 0$, as

$$L(s, \mathbf{f}, \chi) = \sum_{\mathfrak{b}} \frac{\chi(\mathfrak{b})C(\mathfrak{b}, \mathbf{f})}{\mathcal{N}\mathfrak{b}^s} \tag{2.39}$$

(here $\chi(\mathfrak{b}) = 0$ if $\mathfrak{b}$ is not relatively prime to the conductor of χ). By the eigenform property it admits the Euler product over prime ideals:

$$L(s, \mathbf{f}, \chi) = \prod_{\mathfrak{l}} \frac{1}{1 - \chi(\mathfrak{l})C(\mathfrak{l}, \mathbf{f})\mathcal{N}\mathfrak{l}^{-s} + \epsilon_{\mathfrak{l}}\chi(\mathfrak{l})^2\mathcal{N}\mathfrak{l}^{m+1-2s}} \tag{2.40}$$

(here $\epsilon_{\mathfrak{l}} = 0$ if $\mathfrak{l}$ divides $\mathfrak{n}$, and is one otherwise). The function $L(s, \mathbf{f})$ can be analytically continued to an entire function, and admits a functional equation relating the value $L(s, \mathbf{f}, \chi)$ to $L(m+2-s, \mathbf{f}, \chi^{-1})$. If χ is trivial then the value $L(m/2+1, \mathbf{f})$ is the central L-value.

Finally recall a version of the Jacquet–Langlands–Shimizu correspondence, as can be found as Theorem 2.30 in [7] for instance:

Theorem 2.18 *(Jacquet–Langlands–Shimizu correspondence). Suppose that $\mathfrak{n} = \mathfrak{m}\mathfrak{n}^+\mathfrak{n}^-$ as before. Let $\Phi \in S^B_{\underline{n},\underline{v}}(\mathfrak{m}\mathfrak{n}^+, \mathfrak{n}^-)$ be an eigenform. In the case where $\underline{n} = \underline{0}$ (and hence $\underline{v} = \frac{m}{2}\underline{t}$) assume that $\Phi \neq (\chi_{F,\mathrm{cycl}} \circ \mathrm{Nrd}_{\widehat{B}/\widehat{F}})^{-m/2}$. Then there is a unique normalized Hilbert eigenform $\mathbf{f}$ of weight $(\underline{n}+2\underline{t}, \underline{v})$, that is new at primes dividing $\mathfrak{n}^-$, such that the eigenvalues of Φ and $\mathbf{f}$ with respect to $\mathbf{T}$ coincide. Conversely given a Hilbert eigenform $\mathbf{f}$ of weight $(\underline{n} + 2\underline{t}, \underline{v})$, level $\mathfrak{n}$, that is new at primes dividing $\mathfrak{n}^-$, there is an eigenform $\Phi \in S_{\underline{n},\underline{v}}(\mathfrak{m}\mathfrak{n}^+, \mathfrak{n}^-)$, unique up to scalar multiples, such that the eigenvalues of $\mathbf{f}$ and Φ with respect to $\mathbf{T}$ coincide. If Φ is a newform then so is $\mathbf{f}$ and conversely.*

Remark 2.19. With Φ and $\mathbf{f}$ as in Theorem 2.18, let Φ' and $\mathbf{f}'$ be the corresponding twisted forms as in Remark 2.4 and Remark 2.16 respectively (for the same integer r). Then Φ' and $\mathbf{f}'$ correspond under the Jacquet–Langlands–Shimizu correspondence.

3. Teitelbaum type $\mathcal{L}$-invariant

3.1. Rigid analytic modular forms and Coleman integrals

In this section, we define rigid analytic modular forms and Coleman integrals associated to the harmonic cocycles defined by (2.21). One new aspect that occurs when $F \neq \mathbf{Q}$ is that the rigid analytic forms and Coleman integrals to be considered are vector valued rather scalar valued, and that we need to consider all the different embeddings $\sigma \in I_{\mathfrak{p}}$. The definitions are motivated by the Cerednik–Drinfeld theorem on p-adic uniformization of Shimura curves over totally real fields, in the form proved by Varshavsky [17, Theorem 5.3].

For $\mathfrak{p}$ as before, put $\mathcal{W}_{\mathfrak{p}} := F_{\mathfrak{p}}^2 - \{0, 0\}$. Define, for $\sigma \in I_{\mathfrak{p}}$, the natural projection:

$$\begin{aligned} \mathrm{pr}_{\mathfrak{p}}^{\sigma} : \mathcal{W}_{\mathfrak{p}} &\to \mathbf{P}^1\big(F_{\mathfrak{p}}^{\sigma}\big) = F_{\mathfrak{p}}^{\sigma} \cup \{\infty\} \\ \mathrm{pr}_{\mathfrak{p}}^{\sigma}((x, y)) &= (x/y)^{\sigma}. \end{aligned} \tag{3.1}$$

We will usually write t for the affine coordinate of $\mathbf{P}^1(F_{\mathfrak{p}}^{\sigma})$. We let $\mathrm{GL}_2(F_{\mathfrak{p}})$ act on $\mathcal{W}_{\mathfrak{p}}$ by the rule: for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(F_{\mathfrak{p}})$, and $(x, y) \in \mathcal{W}_{\mathfrak{p}}$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (x, y) = (ax + by, cx + dy). \tag{3.2}$$

On the other hand, define the action of $\mathrm{GL}_2(F_{\mathfrak{p}})$ on $\mathbf{P}^1(F_{\mathfrak{p}}^{\sigma})$ by the rule: for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(F_{\mathfrak{p}})$ and $t \in \mathbf{P}^1(F_{\mathfrak{p}}^{\sigma})$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot t = \frac{a^{\sigma} t + b^{\sigma}}{c^{\sigma} t + d^{\sigma}}. \tag{3.3}$$

Then $\mathrm{pr}_{\mathfrak{p}}^{\sigma}$ is equivariant for the action of $\mathrm{GL}_2(F_{\mathfrak{p}})$.

For any lattice L of $F_{\mathfrak{p}}^2$, define $L' := L - \pi_{\mathfrak{p}} L$ to be the set of primitive vectors. Given an oriented edge $e = (s, t) \in \mathcal{E}(\mathcal{T}_{\mathfrak{p}})$, choose lattices L_s and L_t that represent s and t, such that L_s contains L_t with index $\mathcal{N}\mathfrak{p}$. Put

$$U_e^{\sigma} := \mathrm{pr}_{\mathfrak{p}}^{\sigma}\big(L_s' \cap L_t'\big). \tag{3.4}$$

Note that U_e^{σ} is independent of the choices of $L_s \supset L_t$ and depends only on e. It is an open compact subset of $\mathbf{P}^1(F_{\mathfrak{p}}^{\sigma})$, and $\{U_e^{\sigma}\}_{e \in \mathcal{E}(\mathcal{T}_{\mathfrak{p}})}$ forms a base of the p-adic topology of $\mathbf{P}^1(F_{\mathfrak{p}}^{\sigma})$.

Referring to the notations of Section 2.1, for $(\underline{n}, \underline{v})$ as before, put $L^\sigma = L^\sigma(\underline{n}, \underline{v}) := \bigotimes_{\sigma' \neq \sigma} L_{\sigma'}(n_{\sigma'}, v_{\sigma'})$. We identify $L_\sigma(n_\sigma, v_\sigma) \otimes_{\mathbf{C}_p} L^\sigma(\underline{n}, \underline{v}) \cong L(\underline{n}, \underline{v})$. Equip $L^\sigma(\underline{n}, \underline{v})$ with the right action of $B_p^\times$ as in Definition 2.1 (except for the disappearance of the factor $L_\sigma(n_\sigma, v_\sigma)$). Put $V^\sigma(\underline{n}, \underline{v})$ to be the $\mathbf{C}_p$-linear dual of $L^\sigma(\underline{n}, \underline{v})$, with the dual left action of $B_p^\times$.

For Φ satisfying the conditions of Proposition 2.7, we have the $h_\mathfrak{p}$-tuple of harmonic cocycle $c_{\phi_\mathfrak{p}^i}$ defined as in (2.21). We want to define a $V^\sigma = V^\sigma(\underline{n}, \underline{v})$-valued locally analytic distribution $\mu^\sigma_{\phi_\mathfrak{p}^i}$ on $\mathbf{P}^1(F_\mathfrak{p}^\sigma)$ associated to $c_{\phi_\mathfrak{p}^i}$ for $i = 1, \cdots, h_\mathfrak{p}$.

In general let $c \in C^1_{\mathrm{har}}(\mathcal{T}_\mathfrak{p}, V)^{\Gamma_i^\mathfrak{p}}$. As described in [16, Proposition 9], the methods of Amice–Velu and Vishik allow us to define the V^σ-valued locally analytic distribution μ_c^σ on $\mathbf{P}^1(F_\mathfrak{p}^\sigma)$, characterized by the property: for any $P \in L_\sigma(n_\sigma, v_\sigma)$, the value of the integral

$$\int_{U_e^\sigma} P(t)\, d\mu_c^\sigma \in V^\sigma$$

is given by the following: for any $Q \in L^\sigma$,

$$\left(\int_{U_e^\sigma} P(t)\, d\mu_c^\sigma(t)\right)(Q) = c_{\phi_\mathfrak{p}^i}(e)(P \otimes_{\mathbf{C}_p} Q). \tag{3.5}$$

The distribution μ_c^σ can be used to integrate locally meromorphic functions on $\mathbf{P}^1(F_\mathfrak{p}^\sigma)$ that are locally analytic on $F_\mathfrak{p}^\sigma$, and with a pole of order at most n_σ at ∞.

Note that by condition (2.19), we have

$$\int_{\mathbf{P}^1(F_\mathfrak{p}^\sigma)} P(t)\, d\mu_c^\sigma(t) = 0 \quad \text{for all } P \in L_\sigma(n_\sigma, v_\sigma). \tag{3.6}$$

The invariance of the harmonic cocycle c with respect to $\widetilde{\Gamma}_i^\mathfrak{p}$ can then be stated as follows: for all $\gamma \in \widetilde{\Gamma}_i^\mathfrak{p}$, and $P \in L_\sigma(n_\sigma, v_\sigma)$,

$$\int_{U^\sigma_{\gamma_\mathfrak{p} e}} P(t)\, d\mu_c^\sigma(t) = |\operatorname{Nrd}_{B/F} \gamma|_\mathfrak{p}^{m/2} \gamma_p \cdot \left(\int_{U_e^\sigma} P|\gamma_\mathfrak{p}(t)\, d\mu_c^\sigma(t)\right). \tag{3.7}$$

Let $\mathcal{H}_{F_\mathfrak{p}^\sigma}$ be the p-adic upper half space over $F_\mathfrak{p}^\sigma$, which is a rigid analytic space over $F_\mathfrak{p}^\sigma$ whose $\mathbf{C}_p$-points are given by $\mathcal{H}_{F_\mathfrak{p}^\sigma}(\mathbf{C}_p) = \mathbf{P}^1(\mathbf{C}_p) - \mathbf{P}^1(F_\mathfrak{p}^\sigma)$; notice that the embedding σ of $F_\mathfrak{p}$ into $\mathbf{C}_p$ has to be specified (if $F_\mathfrak{p}/\mathbf{Q}_p$ is Galois, for example when $\mathfrak{p}|p$ is unramified, then $\mathcal{H}_{F_\mathfrak{p}^\sigma}$ is independent of the choice of $\sigma \in I_\mathfrak{p}$). We equipped the action of $\mathrm{GL}_2(F_\mathfrak{p})$ on $\mathcal{H}_{F_\mathfrak{p}^\sigma}$ as in the case of $\mathbf{P}^1(F_\mathfrak{p}^\sigma)$, *cf.* Eq. (3.3).

Denote by $\mathcal{O}_{\mathcal{H}_{F_\mathfrak{p}^\sigma}}$ the structure sheaf of $\mathbf{C}_p$-algebras of the rigid space $\mathcal{H}_{F_\mathfrak{p}^\sigma}$. Define the V^σ-valued rigid analytic function f_c^σ on $\mathcal{H}_{F_\mathfrak{p}^\sigma}(\mathbf{C}_p)$ by the p-adic Poisson type integral formula

$$f_c^\sigma(z) := \int_{\mathbf{P}^1(F_\mathfrak{p}^\sigma)} \frac{1}{t-z}\, d\mu_c^\sigma(t) \quad \text{for } z \in \mathcal{H}_{F_\mathfrak{p}^\sigma}(\mathbf{C}_p). \tag{3.8}$$

Using (3.6) and (3.7), one can show that the function f_c^σ satisfies the transformation property: for $\gamma \in \widetilde{\Gamma}_i^\mathfrak{p}$, let $\gamma^\sigma = \left(\begin{smallmatrix} a^\sigma & b^\sigma \\ c^\sigma & d^\sigma \end{smallmatrix}\right) \in \mathrm{GL}_2(F_\mathfrak{p}^\sigma)$ be the image of $\gamma_\mathfrak{p} \in \mathrm{GL}_2(F_\mathfrak{p})$ under σ. Then:

$$f_c^\sigma(\gamma_\mathfrak{p}\cdot z) = |\operatorname{Nrd}_{B/F}\gamma|_\mathfrak{p}^{m/2}\left(\operatorname{Nrd}_{B/F}\gamma^\sigma\right)^{v_\sigma-1}\left(c^\sigma z+d^\sigma\right)^{n_\sigma+2}\gamma_p\cdot\left(f_c^\sigma(z)\right). \tag{3.9}$$

The V^σ-valued function f_c^σ is an example of a vector-valued rigid analytic modular form.

We now define the Coleman integrals. For $\tau_1, \tau_2 \in \mathcal{H}_{F_\mathfrak{p}^\sigma}$, and $P \in L_\sigma(n_\sigma, v_\sigma)$, define

$$\int_{\tau_1}^{\tau_2} P(z) f_{\phi_\mathfrak{p}^i}^\sigma(z)dz := \int_{\mathbf{P}^1(F_\mathfrak{p}^\sigma)} P(t)\log_p\left(\frac{t-\tau_2}{t-\tau_1}\right) d\mu_c^\sigma(t) \in V^\sigma. \tag{3.10}$$

Here $\log_p : \mathbf{C}_p^\times \to \mathbf{C}_p$ is Iwasawa's p-adic logarithm, defined by the condition $\log_p(p) = 0$.

Proposition 3.1. *For any $\gamma \in \widetilde{\Gamma}_i^\mathfrak{p}$, we have:*

$$\int_{\gamma_\mathfrak{p}\tau_1}^{\gamma_\mathfrak{p}\tau_2} P(z) f_c^\sigma(z)dz = |\operatorname{Nrd}_{B/F}\gamma|_\mathfrak{p}^{m/2}\gamma_p\cdot\left(\int_{\tau_1}^{\tau_2}(P|\gamma_\mathfrak{p})(z) f_c^\sigma(z)dz\right).$$

Proof. With notations as in (3.9) we compute, using (3.7):

$$\begin{aligned}
\int_{\gamma_\mathfrak{p}\tau_1}^{\gamma_\mathfrak{p}\tau_2} P(z) f_c^\sigma(z)dz &= \int_{\mathbf{P}^1(F_\mathfrak{p}^\sigma)} P(t)\log_p\left(\frac{t-\gamma_\mathfrak{p}\tau_2}{t-\gamma_\mathfrak{p}\tau_1}\right) d\mu_c^\sigma(t) \\
&= |\operatorname{Nrd}_{B/F}\gamma|_\mathfrak{p}^{m/2}\gamma_p\cdot\left(\int_{\mathbf{P}^1(F_\mathfrak{p}^\sigma)} P|\gamma_\mathfrak{p}(t)\log_p\left(\frac{\gamma_\mathfrak{p}t-\gamma_\mathfrak{p}\tau_2}{\gamma_\mathfrak{p}t-\gamma_\mathfrak{p}\tau_1}\right) d\mu_c^\sigma(t)\right) \\
&= |\operatorname{Nrd}_{B/F}\gamma|_\mathfrak{p}^{m/2}\gamma_p\cdot\left(\int_{\mathbf{P}^1(F_\mathfrak{p}^\sigma)} P|\gamma_\mathfrak{p}(t)\log_p\left(\frac{t-\tau_2}{t-\tau_1}\frac{c^\sigma\tau_1+d^\sigma}{c^\sigma\tau_2+d^\sigma}\right) d\mu_c^\sigma(t)\right) \\
&= |\operatorname{Nrd}_{B/F}\gamma|_\mathfrak{p}^{m/2}\gamma_p\cdot\left(\int_{\mathbf{P}^1(F_\mathfrak{p}^\sigma)} P|\gamma_\mathfrak{p}(t)\log_p\left(\frac{t-\tau_2}{t-\tau_1}\right) d\mu_c^\sigma(t)\right) \\
&\quad + |\operatorname{Nrd}_{B/F}\gamma|_\mathfrak{p}^{m/2}\log_p\left(\frac{c^\sigma\tau_1+d^\sigma}{c^\sigma\tau_2+d^\sigma}\right)\gamma_p\cdot\left(\int_{\mathbf{P}^1(F_\mathfrak{p}^\sigma)} P|\gamma_\mathfrak{p}(t)\, d\mu_c^\sigma(t)\right).
\end{aligned}$$

By (3.6), the second term in the last expression is zero, and hence we obtain the result. □

3.2. Definition of $\mathcal{L}$-invariants

With the notation of the previous section, let $c \in C^1_{\mathrm{har}}(\mathcal{T}_{\mathfrak{p}}, V)^{\Gamma_i^{\mathfrak{p}}}$. For each $\sigma \in I_{\mathfrak{p}}$, define a V-valued cochain λ_c^σ on $\Gamma_i^{\mathfrak{p}}$ as follows. Choose base point $z_0 \in \mathcal{H}_{F_{\mathfrak{p}}^\sigma}$. Let $\gamma \in \widetilde{\Gamma}_i^{\mathfrak{p}}$, and by abuse of notation we continue to denote its image in $\Gamma_i^{\mathfrak{p}}$ as γ. The value $\lambda_c^\sigma(\gamma) \in V$ is determined as follows: it suffices to specify the evaluation of $\lambda_c^\sigma(\gamma)$ on tensors of the form $P \otimes Q \in L = L(\underline{n}, \underline{v})$, with $P \in L_\sigma(n_\sigma, v_\sigma)$, $Q \in L^\sigma(\underline{n}, \underline{v})$; then

$$\lambda_c^\sigma(\gamma)(P \otimes Q) = \left(\int_{z_0}^{\gamma_{\mathfrak{p}} z_0} P(z) f_c^\sigma(z) dz \right)(Q). \tag{3.11}$$

By (3.11) it can be checked that λ_c^σ is a cocycle on $\Gamma_i^{\mathfrak{p}}$ (with respect to the $\star_{\mathfrak{p}}$ action of $\Gamma_i^{\mathfrak{p}}$ on V as in Section 2.4), and that the class of λ_c^σ in $H^1(\Gamma_i^{\mathfrak{p}}, V)$ is independent of the choice of z_0. This defines a map

$$\begin{aligned} \kappa_{i,\mathfrak{p}}^{\mathrm{col},\sigma} : C^1_{\mathrm{har}}(\mathcal{T}_{\mathfrak{p}}, V)^{\Gamma_i^{\mathfrak{p}}} &\to H^1\left(\Gamma_i^{\mathfrak{p}}, V\right) \\ c &\mapsto \text{class of } \lambda_c^\sigma. \end{aligned} \tag{3.12}$$

Denote by $\kappa_{\mathfrak{p}}^{\mathrm{col},\sigma}$ the direct sum of the maps

$$\kappa_{\mathfrak{p}}^{\mathrm{col},\sigma} : \bigoplus_{i=1}^{h_{\mathfrak{p}}} C^1_{\mathrm{har}}(\mathcal{T}_{\mathfrak{p}}, V)^{\Gamma_i^{\mathfrak{p}}} \to \bigoplus_{i=1}^{h_{\mathfrak{p}}} H^1\left(\Gamma_i^{\mathfrak{p}}, V\right). \tag{3.13}$$

The same computations as in Section 2.5 show that

Proposition 3.2. *The map $\kappa_{\mathfrak{p}}^{\mathrm{col},\sigma}$ commutes the Hecke operators $T_{\mathfrak{l}}, U_{\mathfrak{l}}$ for all $\mathfrak{l} \neq \mathfrak{p}$.*

Let $\Phi \in S_{\underline{n},\underline{v}}(\mathfrak{m}\mathfrak{n}^+, \mathfrak{n}^-)$ be a newform that satisfies the condition of Proposition 2.7, and corresponds to the tuple $(\phi_{\mathfrak{p}}^1, \cdots, \phi_{\mathfrak{p}}^{h_{\mathfrak{p}}})$. For $\sigma \in I_{\mathfrak{p}}$, we have the cohomology class $\kappa_{i,\mathfrak{p}}^{\mathrm{col},\sigma}(\phi_{\mathfrak{p}}^i)$ given by the Coleman integral. Denote $\kappa_{\mathfrak{p}}^{\mathrm{col},\sigma}(\Phi) = \bigoplus_{i=1}^{h_{\mathfrak{p}}} \kappa_{i,\mathfrak{p}}^{\mathrm{col},\sigma}(\phi_{\mathfrak{p}}^i)$. Thus we have a pair of $h_{\mathfrak{p}}$-tuple of cohomology class $\kappa_{\mathfrak{p}}^{\mathrm{col},\sigma}(\Phi), \kappa_{\mathfrak{p}}^{\mathrm{sch},\sigma}(\Phi)$.

To define the Teitelbaum type $\mathcal{L}$-invariant, we need the following multiplicity one statement.

Proposition 3.3. *The Φ-eigenspace of the module $\bigoplus_{i=1}^{h_{\mathfrak{p}}} H^1(\Gamma_i^{\mathfrak{p}}, V)$ with respect to the operators $T_{\mathfrak{l}}, U_{\mathfrak{l}}$ for all $\mathfrak{l} \neq \mathfrak{p}$ is one-dimensional.*

Proof. By Proposition 2.9 we have an isomorphism between $\bigoplus_{i=1}^{h_{\mathfrak{p}}} H^1(\Gamma_i^{\mathfrak{p}}, V)$ and $\bigoplus_{i=1}^{h_{\mathfrak{p}}} C^1_{\mathrm{har}}(\mathcal{T}_{\mathfrak{p}}, V)^{\Gamma_i^{\mathfrak{p}}}$, and the latter is isomorphic to the $U_{\mathfrak{p}}$-eigenspace of

$S^B_{\underline{n},\underline{v}}(\mathfrak{m}\mathfrak{n}^+,\mathfrak{n}^-)^{\mathfrak{p}-new}$ with the $U_\mathfrak{p}$-eigenvalue being equal to $\mathcal{N}\mathfrak{p}^{m/2}$; the isomorphisms commute with the Hecke operators $T_\mathfrak{l}, U_\mathfrak{l}$ for all $\mathfrak{l} \neq \mathfrak{p}$. Thus the Φ-eigenspace of the module $\bigoplus_{i=1}^{h_\mathfrak{p}} H^1(\Gamma_i^\mathfrak{p}, V)$ is isomorphic to the Φ-eigenspace of $S^B_{\underline{n},\underline{v}}(\mathfrak{m}\mathfrak{n}^+,\mathfrak{n}^-)^{\mathfrak{p}-new}$ with respect to all $T_\mathfrak{l}, U_\mathfrak{l}$. This is one-dimensional by Theorem 2.18. □

Thus both $\kappa_\mathfrak{p}^{\mathrm{col},\sigma}(\Phi)$ and $\kappa_\mathfrak{p}^{\mathrm{sch}}(\Phi)$ lie in Φ-isotypic component of the Hecke module $\bigoplus_{i=1}^{h_\mathfrak{p}} H^1(\Gamma_i^\mathfrak{p}, V)$. By Proposition 3.3, the Φ-isotypic component of $\bigoplus_{i=1}^{h_\mathfrak{p}} H^1(\Gamma_i^\mathfrak{p}, V)$ is one-dimensional, with a basis given by $\kappa_\mathfrak{p}^{\mathrm{sch}}(\Phi)$. It follows that there is a unique $\mathcal{L}_\mathfrak{p}^{\sigma,\mathrm{Tei}}(\Phi) \in \mathbf{C}_p$, such that

$$\kappa_\mathfrak{p}^{\mathrm{col},\sigma}(\Phi) = \mathcal{L}_\mathfrak{p}^{\sigma,\mathrm{Tei}}(\Phi)\kappa_\mathfrak{p}^{\mathrm{sch}}(\Phi). \tag{3.14}$$

In particular we have

$$\kappa_{i,\mathfrak{p}}^{\mathrm{col},\sigma}\left(\phi_\mathfrak{p}^i\right) = \mathcal{L}_\mathfrak{p}^{\sigma,\mathrm{Tei}}(\Phi)\kappa_{i,\mathfrak{p}}^{\mathrm{sch}}\left(\phi_\mathfrak{p}^i\right) \tag{3.15}$$

for all $i = 1, \cdots, h_\mathfrak{p}$.

Definition 3.4. Let Φ satisfy the condition of Proposition 2.7, and $\mathbf{f}$ be the Hilbert newform corresponding to Φ under the Jacquet–Langlands–Shimizu correspondence. The Teitelbaum $\mathcal{L}$ invariant $\mathcal{L}_\mathfrak{p}^{\sigma,\mathrm{Tei}}(\mathbf{f})$ of $\mathbf{f}$ at the prime $\mathfrak{p}$, with respect to $\sigma \in I_\mathfrak{p}$, is the quantity $\mathcal{L}_\mathfrak{p}^{\sigma,\mathrm{Tei}}(\Phi)$ as defined in (3.15) (recall that $\mathbf{f}$ determines Φ uniquely up to scalar multiple, so $\mathcal{L}_\mathfrak{p}^{\sigma}(\mathbf{f})$ is well-defined). Finally put

$$\mathcal{L}_\mathfrak{p}^{\mathrm{Tei}}(\mathbf{f}) := \sum_{\sigma \in I_\mathfrak{p}} \mathcal{L}_\mathfrak{p}^{\sigma,\mathrm{Tei}}(\mathbf{f}). \tag{3.16}$$

It can be checked that the $\mathcal{L}$-invariant $\mathcal{L}_\mathfrak{p}^{\sigma,\mathrm{Tei}}(\mathbf{f})$ is independent of the choices that occur in (2.1), (2.2) and (2.15). On the other hand, the reason that we define the quantity $\mathcal{L}_\mathfrak{p}^{\mathrm{Tei}}(\mathbf{f})$ as the sum of $\mathcal{L}_\mathfrak{p}^{\sigma,\mathrm{Tei}}(\mathbf{f})$ over $\sigma \in I_\mathfrak{p}$, is that it is the quantity $\mathcal{L}_\mathfrak{p}^{\mathrm{Tei}}(\mathbf{f})$ that is expected to appear in the general form of the exceptional zero conjecture; see Conjecture 4.2 below, and also Remark 4.3 for the justification of taking the sum over $\sigma \in I_\mathfrak{p}$.

Remark 3.5. With $\mathbf{f}$ as above, let $\mathbf{f}'$ be the twist of $\mathbf{f}$ as in Remark 2.16. It follows from Remark 2.19 and Remark 2.8 that $\mathcal{L}_\mathfrak{p}^{\sigma,\mathrm{Tei}}(\mathbf{f}) = \mathcal{L}_\mathfrak{p}^{\sigma,\mathrm{Tei}}(\mathbf{f}')$.

With $\mathbf{f}$ as in Definition 3.4, we have that $\mathfrak{p}$ divides exactly the conductor of $\mathbf{f}$, and that

$$C(\mathfrak{p}, \mathbf{f}) = \mathcal{N}\mathfrak{p}^{m/2}.$$

This condition is equivalent to saying that the $\mathfrak{p}$-component of the cuspidal automorphic representation generated by $\mathbf{f}$ is the special representation of $\mathrm{GL}_2(F_\mathfrak{p})$ (by the results of Casselman [3]). Therefore we can generalize the Fontaine–Mazur type $\mathcal{L}$-invariant $\mathcal{L}_{\mathfrak{p}}^{\sigma,\mathrm{FM}}(\mathbf{f})$ for each $\sigma \in I_{\mathfrak{p}}$ as follows.

Let $\mathbf{f}$ be a Hilbert eigenform, whose weight is $(\underline{k}, \underline{v})$ (here $\underline{k} = (k_\sigma)_\sigma$, $\underline{v} = (v_\sigma)_\sigma$ with $k_\sigma \in \mathbf{Z}^{\geq 2}$, $v_\sigma \in \mathbf{Z}$, and σ ranges over the set of embeddings of F into $\overline{\mathbf{Q}}_p$, as in Definition 2.14). Let G_F be the absolute Galois group of F, and denote by

$$\rho_{\mathbf{f}} : G_F \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$$

the two-dimensional p-adic Galois representation associated to the eigenform Hilbert $\mathbf{f}$, characterized by the condition

$$\mathrm{Tr}\, \rho_{\mathbf{f}}(\mathrm{Frob}_{\mathfrak{l}}) = C(\mathfrak{l}, \mathbf{f})$$

for all primes $\mathfrak{l}$ of F not dividing p and the conductor of $\mathbf{f}$; here $\mathrm{Frob}_{\mathfrak{l}}$ is the Frobenius element at $\mathfrak{l}$ and $C(\mathfrak{l}, \mathbf{f})$ is the normalized Fourier coefficient of $\mathbf{f}$ at $\mathfrak{l}$, as in Eq. (2.36) below. Since we assume that $\mathbf{f}$ is special at the prime $\mathfrak{p}$, the Galois representation $\rho_{\mathbf{f}}$ was already constructed by Carayol [2].

Put $\rho := \rho_{\mathbf{f}}|_{G_{F_{\mathfrak{p}}}}$, the restriction of $\rho_{\mathbf{f}}$ to the local Galois group $G_{F_{\mathfrak{p}}}$ at $\mathfrak{p}$. Denote by $D_{\mathrm{st}}(\rho)$ the semi-stable Dieudonne module of Fontaine associated to ρ. From the main result of [12], we have the following information about $D_{\mathrm{st}}(\rho)$. It is free of rank two over $F_{\mathfrak{p},0} \otimes_{\mathbf{Q}_p} \overline{\mathbf{Q}}_p$ (where $F_{\mathfrak{p},0}$ is the maximal subextension of $F_{\mathfrak{p}}/\mathbf{Q}_p$ that is absolutely unramified). Put $D := D_{\mathrm{st}}(\rho) \otimes_{F_{\mathfrak{p},0}} F_{\mathfrak{p}}$. Then D is free of rank two over $F_{\mathfrak{p}} \otimes_{\mathbf{Q}_p} \overline{\mathbf{Q}}_p$.

For any $\sigma : F_{\mathfrak{p}} \to \overline{\mathbf{Q}}_p$ it induces the map $F_{\mathfrak{p}} \otimes_{\mathbf{Q}_p} \overline{\mathbf{Q}}_p \to \overline{\mathbf{Q}}_p$. By [12] the module $D_\sigma := D \otimes_{F_{\mathfrak{p}} \otimes_{\mathbf{Q}_p} \overline{\mathbf{Q}}_p, \sigma} \overline{\mathbf{Q}}_p$ is then a two-dimensional filtered (φ, N)-module over $\overline{\mathbf{Q}}_p$, with the monodromy operator N being non-trivial. The data defining the filtration of D_σ is given by a one-dimensional subspace $F_\sigma \subset D_\sigma$, where the jumps of the filtration are given by $(v_\sigma, k_\sigma + v_\sigma - 1)$ (the underlying (φ, N) module of D_σ is independent of the embedding σ up to isomorphism). Choose a pair of φ-eigenvectors $u_1, u_2 \in D_\sigma$ such that $N(u_1) = u_2$. Then $\{u_1, u_2\}$ is a basis of D_σ, and the Fontaine–Mazur $\mathcal{L}$-invariant $\mathcal{L} = \mathcal{L}_{\mathfrak{p}}^{\sigma,\mathrm{FM}}(\mathbf{f}) \in \overline{\mathbf{Q}}_p$ is the unique element such that

$$F_\sigma = \overline{\mathbf{Q}}_p(u_1 - \mathcal{L} u_2)$$

(the weak admissibility condition satisfied by D_σ insures that $F_\sigma \neq \overline{\mathbf{Q}}_p u_2$). In particular, there is one such $\mathcal{L}$-invariant for each embedding $\sigma : F_{\mathfrak{p}} \to \overline{\mathbf{Q}}_p$. Note that the Fontaine–Mazur type $\mathcal{L}$-invariant $\mathcal{L}_{\mathfrak{p}}^{\sigma,\mathrm{FM}}(\mathbf{f})$ depends only on $\rho_{\mathbf{f}}|_{G_{F_{\mathfrak{p}}}}$.

Conjecture 3.6. *We have the equality*

$$\mathcal{L}_{\mathfrak{p}}^{\sigma,\mathrm{FM}}(\mathbf{f}) = \mathcal{L}_{\mathfrak{p}}^{\sigma,\mathrm{Tei}}(\mathbf{f}).$$

In the case $F = \mathbf{Q}$ this was proved by Iovita and Spiess [8, Theorem 6.4]. Conjecture 3.6 would imply that $\mathcal{L}_{\mathfrak{p}}^{\sigma,\mathrm{FM}}(\mathbf{f})$ can be computed from the automorphic side, and conversely that $\mathcal{L}_{\mathfrak{p}}^{\sigma,\mathrm{Tei}}(\mathbf{f})$ depends only on $\rho_{\mathbf{f}}|_{G_{F_{\mathfrak{p}}}}$. In particular $\mathcal{L}_{\mathfrak{p}}^{\sigma,\mathrm{Tei}}(\mathbf{f})$ does not depend on the factorization of the conductor $\mathfrak{n} = \mathfrak{m}\mathfrak{n}^+\mathfrak{n}^-$.

4. Statement of the exceptional zero conjecture

To conclude, we state the exceptional zero conjecture for the p-adic L-functions of Hilbert modular forms, using the Teitelbaum type $\mathcal{L}$-invariants. Thus in contrast to the previous sections, we do not assume that the Hilbert modular forms arise from definite quaternion algebras via the Jacquet–Langlands–Shimizu correspondence.

We recall the properties of p-adic L-functions of Hilbert modular forms, as defined by Dabrowski [4]. First we need some notations. Let $\psi = \bigotimes_\nu \psi_\nu$ be a Hecke character of F of finite order. Denote by $\mathrm{sig}(\psi) \in \{\pm 1\}^d$, the signature of ψ, as the d-tuple $(\psi_\nu(-1))_{\nu|\infty}$. Thus $\mathrm{sig}(\psi) = (1, \cdots, 1)$ if ψ is unramified at all the infinite places. As another example, let $\omega_{\mathbf{Q}}$ be the Teichmuller character of $\mathbf{Q}$. Regarding $\omega_{\mathbf{Q}}$ as a Hecke character of $\mathbf{Q}$, and letting $\omega_F = \omega_{\mathbf{Q}} \circ \mathcal{N}_{F/\mathbf{Q}}$, one has $\mathrm{sig}(\omega_F) = (-1, \cdots, -1)$.

Denote by $\mathfrak{c}_\psi$ the conductor of ψ, and by $\tau(\psi)$ the Gauss sum associated to ψ [14, Eq. (3.9)].

Let $\mathbf{f} \in S_{\underline{n}+2\underline{t},\underline{v}}(\mathfrak{n})$ be a normalized Hilbert newform, with $\underline{n} + 2\underline{v} = m\underline{t}$ for some even integer m. Put

$$r_* = \max_{\sigma\in I} v_\sigma, \qquad r^* = \min_{\sigma\in I}(n_\sigma + v_\sigma) = m - r_*.$$

Then for $r_* \le r \le r^*$, the values $L(r+1, \mathbf{f})$ are the critical values in the sense of Deligne of the complex L-function $L(s, \mathbf{f})$ associated to f. The central critical value is $L(m/2 + 1, \mathbf{f})$.

We have Shimura's rationality result on L-values, *cf.* [14, Theorem 4.3(I)], and [4, remark (ii) on p. 1027]: for every $w \in \{\pm 1\}^d$, one can choose $\Omega_{\mathbf{f}}^w \in \mathbf{C}^\times$, such that, for integer $r_* \le r \le r^*$, and finite order Hecke character ψ of F, with conductor $\mathfrak{c}_\psi$, the value:

$$L^{\mathrm{alg}}(r+1, \mathbf{f}, \psi) := \frac{\prod_{\sigma\in I} \Gamma(r+1-v_\sigma)}{\prod_{\sigma\in I}(-2\pi i)^{r-v_\sigma}} \cdot \frac{D_F^r L(r+1, \mathbf{f}, \psi)}{\tau(\psi^{-1})\Omega_{\mathbf{f}}^{(-1)^r \mathrm{sig}(\psi)}} \tag{4.1}$$

lies in the (finite) field extension of $\mathbf{Q}$ generated by the normalized Fourier coefficients of $\mathbf{f}$ and the values of ψ, so in particular is an algebraic number (here D_F is the discriminant of F, and $\Gamma(s)$ is Euler's Γ-function). It is called the algebraic part of the critical value $L(r+1, \mathbf{f}, \psi)$.

Now for each $\mathfrak{q}|p$, factor the Hecke polynomial at $\mathfrak{q}$:

$$X^2 - C(\mathfrak{q}, \mathbf{f})X + \epsilon_{\mathfrak{q}}\mathcal{N}\mathfrak{q}^{m+1} = \big(X - \alpha(\mathfrak{q})\big)\big(X - \beta(\mathfrak{q})\big) \tag{4.2}$$

(here $\epsilon_{\mathfrak{q}} = 1$ if $\mathfrak{q} \nmid \mathfrak{n}$, and equal to zero otherwise).

For each $\mathfrak{q}|p$ order the roots $\alpha(\mathfrak{q}), \beta(\mathfrak{q})$ so that $\mathrm{ord}_p\, \alpha(\mathfrak{q}) \le \mathrm{ord}_p\, \beta(\mathfrak{q})$. Assume that $\mathrm{ord}_p\, \alpha(\mathfrak{q}) < \infty$ for all $\mathfrak{p}|p$. We refer to [4, Theorem 1] for the following statement:

Theorem 4.1. *For any finite order Hecke character χ of F unramified outside the places p and ∞, there is a p-adic analytic function $L_p(s, \mathbf{f}, \chi)$ for $s \in \mathbf{Z}_p$ (called the p-adic L-function of $\mathbf{f}$ and χ), that satisfies the following interpolation property: for all integers $r_* \leq r \leq r^*$, we have*

$$L_p(r+1, \mathbf{f}, \chi) = \prod_{\mathfrak{q}|p} \mathcal{E}_{\mathfrak{q}}(\mathbf{f}, \chi, r) \cdot L^{\mathrm{alg}}\big(r+1, \mathbf{f}, \big(\chi\omega_F^{-r}\big)^{-1}\big). \tag{4.3}$$

Here

$$\mathcal{E}_{\mathfrak{q}}(\mathbf{f}, \chi, r) = \left(1 - \frac{\chi\omega_F^{-r}(\mathfrak{q})\mathcal{N}\mathfrak{q}^r}{\alpha(\mathfrak{q})}\right)\left(1 - \frac{(\chi\omega_F^{-r})^{-1}(\mathfrak{q})\beta(\mathfrak{q})}{\mathcal{N}\mathfrak{q}^{r+1}}\right) \quad \text{if } \mathfrak{q} \nmid \mathfrak{c}_{\chi\omega_F^{-r}}. \tag{4.4}$$

On the other hand if $\mathfrak{q}|\mathfrak{c}_{\chi\omega_F^{-r}}$, then

$$\mathcal{E}_{\mathfrak{q}}(\mathbf{f}, \chi, r) = \left(\frac{\mathcal{N}\mathfrak{q}^{r+1}}{\alpha(\mathfrak{q})}\right)^n \quad \text{with } n = \mathrm{val}_{\mathfrak{q}} \mathfrak{c}_{\chi\omega_F^{-r}}. \tag{4.5}$$

Let S be the set of primes $\mathfrak{p}|p$ such that $\mathfrak{p}$ exactly divides $\mathfrak{n}$, and such that

$$\alpha(\mathfrak{p}) = C(\mathfrak{p}, \mathbf{f}) = \mathcal{N}\mathfrak{p}^{m/2}. \tag{4.6}$$

If $\mathfrak{p} \in S$, and χ_0 is a finite order Hecke character of F such that $\chi_0(\mathfrak{p}) = 1$, then by (4.4), we have $\mathcal{E}_{\mathfrak{p}}(\mathbf{f}, \chi_0\omega_F^{m/2}, m/2) = 0$. Hence in this case, we have by (4.4):

$$L_p\big(m/2+1, \mathbf{f}, \chi_0\omega_F^{m/2}\big) = 0 \tag{4.7}$$

in which case we say that $L_p(s, \mathbf{f}, \chi_0\omega_F^{m/2})$ has an *exceptional zero* at $s = m/2 + 1$.

Now we state

Conjecture 4.2. *Denote by e the cardinality of S. Assume that $e \geq 1$. Then for χ_0 a finite order Hecke character such that $\chi_0(\mathfrak{p}) = 1$ for all $\mathfrak{p} \in S$, the p-adic L-function $L_p(s, \mathbf{f}, \chi_0\omega_F^{m/2})$ vanishes to order at least e at $s = m/2+1$. We also have the derivative formula:*

$$\begin{aligned}&\frac{d^e}{ds^e} L_p\big(s, \mathbf{f}, \chi_0\omega_F^{m/2}\big)\Big|_{s=m/2+1} \\ &\quad = \prod_{\mathfrak{p}\in S} \mathcal{L}_{\mathfrak{p}}^{\mathrm{Tei}}(\mathbf{f}) \prod_{\mathfrak{q}|p,\ \mathfrak{q}\notin S} \mathcal{E}_{\mathfrak{q}}\big(\mathbf{f}, \chi_0\omega_F^{m/2}, m/2\big) \cdot L^{\mathrm{alg}}(m/2+1, \mathbf{f}, \chi_0).\end{aligned} \tag{4.8}$$

Remark 4.3. Let E/F be a modular elliptic curve over F, in the sense that there is a Hilbert newform $\mathbf{f}_E$ of parallel weight 2 with conductor $\mathfrak{n}$ equal to that of E, such that the Galois representation of G_F on the p-adic Tate-module of E/F is isomorphic to $\rho_{\mathbf{f}_E}$. Suppose that $\mathfrak{p} \in S$. Then E has split multiplicative reduction at $\mathfrak{p}$, hence $E/F_{\mathfrak{p}}$ has a

p-adic Tate uniformization. Let $Q_E \in \mathfrak{p}\mathcal{O}_{F_\mathfrak{p}} - \{0\}$ be the Tate period of $E/F_\mathfrak{p}$. Then it can be shown (as in [1, Section II.4]) that for $\sigma \in I_\mathfrak{p}$

$$\mathcal{L}_\mathfrak{p}^{\sigma,\mathrm{FM}}(\mathbf{f}_E) = \frac{\log_p Q_E^\sigma}{\mathrm{val}_\mathfrak{p} Q_E^\sigma} = \frac{\log_p Q_E^\sigma}{\mathrm{val}_\mathfrak{p} Q_E}.$$

Thus

$$\begin{aligned}\mathcal{L}_\mathfrak{p}^{\mathrm{FM}}(\mathbf{f}_E) &:= \sum_{\sigma\in I_\mathfrak{p}} \mathcal{L}_\mathfrak{p}^{\sigma,\mathrm{FM}} \\ &= \sum_{\sigma\in I_\mathfrak{p}} \frac{\log_p Q_E^\sigma}{\mathrm{val}_\mathfrak{p} Q_E} = \frac{\log_p \mathcal{N}_{F_\mathfrak{p}/\mathbf{Q}_p} Q_E}{\mathrm{val}_\mathfrak{p} Q_E} \\ &= f_{\mathfrak{p}/p} \frac{\log_p \mathcal{N}_{F_\mathfrak{p}/\mathbf{Q}_p} Q_E}{\mathrm{ord}_p \mathcal{N}_{F_\mathfrak{p}/\mathbf{Q}_p} Q_E}\end{aligned}$$

(here $f_{\mathfrak{p}/p}$ is the residue field degree of $F_\mathfrak{p}/\mathbf{Q}_p$). So in this case Conjecture 4.2 is the conjecture of Greenberg and Hida (stated as Conjecture 9.1 of [11]) if Conjecture 3.6 holds.

Remark 4.4. Recently Spiess [15] proved the exceptional zero conjecture for modular elliptic curves over totally real fields under certain conditions. In [15], Spiess introduced the automorphic $\mathcal{L}$-invariant $\mathcal{L}_\mathfrak{p}^{\mathrm{Sp}}$ which is a generalization of Darmon's $\mathcal{L}$-invariant defined in [5] and showed the exceptional zero conjecture for Hilbert modular forms with the automorphic $\mathcal{L}$-invariant using a new construction of p-adic L-functions attached to cohomology classes. The coincidence of $\mathcal{L}$-invariant $\mathcal{L}_\mathfrak{p}^{\mathrm{Sp}} = \mathcal{L}_\mathfrak{p}(E)$ is obtained by comparing the exceptional zero formula and the main result of [11].

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Appendix A

Here we prove Proposition 2.9. We largely follow de Shalit [13] whose arguments we extract and suitably generalize.

Recall that $\Gamma_i^{\mathfrak{p}} = \widetilde{\Gamma}_i^{\mathfrak{p}}/(\mathcal{O}_F^{(\mathfrak{p})})^\times$. Put $(\widetilde{\Gamma}_i^{\mathfrak{p}})_1 := \widetilde{\Gamma}_i^{\mathfrak{p}} \cap B_1$, where $B_1 = \{\gamma \in B,\ \mathrm{Nrd}_{B/F}\,\gamma = 1\}$, and $(\Gamma_i^{\mathfrak{p}})_1 := (\widetilde{\Gamma}_i^{\mathfrak{p}})_1/\{\pm 1\}$. Then we have an exact sequence

$$1 \to (\Gamma_i^{\mathfrak{p}})_1 \to \Gamma_i^{\mathfrak{p}} \xrightarrow{\mathrm{Nrd}_{B/F}} (\mathcal{O}_F^{(\mathfrak{p})})^\times/((\mathcal{O}_F^{(\mathfrak{p})})^\times)^2.$$

Note that $(\mathcal{O}_F^{(\mathfrak{p})})^\times/((\mathcal{O}_F^{(\mathfrak{p})})^\times)^2$ is finite, hence $(\Gamma_i^{\mathfrak{q}})_1$ is normal of finite index in $\widetilde{\Gamma}_i^{\mathfrak{p}}$.

Now $\Gamma_i^{\mathfrak{p}}$ embeds as a discrete finitely generated cocompact subgroup of $\mathrm{PGL}_2(F_{\mathfrak{p}})$ [6, Chapter 9], and by [6] we can pick $\Gamma \subset (\Gamma_i^{\mathfrak{p}})_1$ that is normal of finite index in $\Gamma_i^{\mathfrak{p}}$ and is free (called an arithmetic Schottky group). Put $\Delta = \Gamma_i^{\mathfrak{p}}/\Gamma$, then

$$C^1_{\mathrm{har}}(\mathcal{T}_{\mathfrak{p}}, V)^{\Gamma_i^{\mathfrak{p}}} = \left(C^1_{\mathrm{har}}(\mathcal{T}_{\mathfrak{p}}, V)^{\Gamma}\right)^{\Delta}$$
$$H^1\left(\Gamma_i^{\mathfrak{p}}, V\right) = H^1(\Gamma, V)^{\Delta}.$$

Hence it suffices to show that the map

$$\kappa_\Gamma^{\mathrm{sch}} : C^1_{\mathrm{har}}(\mathcal{T}_{\mathfrak{p}}, V)^{\Gamma} \to H^1(\Gamma, V)$$
$$c \to \text{class of } \kappa_c$$

is isomorphism.

Note that since $\Gamma \subset B_1/\{\pm 1\}$, its action on $L(\underline{n}, \underline{v}), V(\underline{n}, \underline{v})$ does not "see" the determinant factor involving $\underline{v}$. In conjunction with this, for an even integer n, denote by $\mathrm{Sym}^n(\mathbf{C}_p^2)$ the $\mathbf{C}_p$-vector space of polynomials in one variable of degree at most n, with right action of $\mathrm{PSL}_2(\mathbf{C}_p) = \mathrm{SL}_2(\mathbf{C}_p)/\{\pm 1\}$ given by the usual rule: for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbf{C}_p)$, and $P(t) \in \mathrm{Sym}^n(\mathbf{C}_p^2)$,

$$P|g(t) = (ct+d)^n \cdot P\left(\frac{at+b}{ct+d}\right).$$

Denote by $(\mathrm{Sym}^n(\mathbf{C}_p^2))^*$ the dual of $\mathrm{Sym}^n(\mathbf{C}_p^2)$, with the dual left action of $\mathrm{SL}_2(\mathbf{C}_p)$.

Next we observe that $\dim C^1_{\mathrm{har}}(\mathcal{T}_{\mathfrak{p}}, V)^{\Gamma} \geq \dim H^1(\Gamma, V)$. Indeed, let $\mathfrak{v}$ and $\mathfrak{e}$ be the number of vertices and unoriented edges of the finite connected graph $\Gamma\backslash\mathcal{T}_{\mathfrak{p}}$. Then $g = \mathfrak{e} - \mathfrak{v} + 1$ is the genus of the graph, and Γ is free on g generators. First compute the dimension of $H^1(\Gamma, V)$. Denote by $D = \prod_{\sigma \in I}(n_\sigma + 1)$ the dimension of $V = V(\underline{n}, \underline{v})$. Then the dimension of the space of cocycles on Γ is gD. On the other hand, the dimension of the space of coboundaries on Γ is equal to D if $\underline{n} \neq \underline{0}$ (since $V^\Gamma = 0$), and equal to 0 if $\underline{n} = \underline{0}$ (since we can take $V = \mathbf{C}_p$). Hence

$$\dim H^1(\Gamma, V) = \begin{cases} (g-1)D & \text{if } \underline{n} \neq \underline{0}, \\ g & \text{if } \underline{n} = \underline{0}. \end{cases}$$

Now consider the dimension of $C^1_{\mathrm{har}}(\mathcal{T}_{\mathfrak{p}}, V)^{\Gamma}$. When $\underline{n} = \underline{0}$, we can take $V = \mathbf{C}_p$, and it's well-known that its dimension is g. In general, to define an element of $C^1_{\mathrm{har}}(\mathcal{T}_{\mathfrak{p}}, V)^{\Gamma}$

one needs to specify an element of V on the $\mathfrak{e}$ unoriented edges of $\Gamma\backslash\mathcal{T}_\mathfrak{p}$, subject to $\mathfrak{v}$ linear conditions at the vertices. It follows that $\dim C^1_{\mathrm{har}}(\mathcal{T}_\mathfrak{p}, V)^\Gamma \geq \mathfrak{e}\cdot D - \mathfrak{v}\cdot D = (g-1)D$, as required.

Thus it suffices to show that $\kappa^{\mathrm{sch}}_\Gamma$ is injective. Denote:

$$C^0(\mathcal{T}_\mathfrak{p}, V) = \{g : \mathcal{V}(\mathcal{T}_\mathfrak{p}) \to V\},$$
$$C^1(\mathcal{T}_\mathfrak{p}, V) = \{f : \mathcal{E}(\mathcal{T}_\mathfrak{p}) \to V \mid f(\bar{e}) = -f(e) \text{ for all edge } e\}.$$

We have a map $d : C^0(\mathcal{T}_\mathfrak{p}, V) \to C^1(\mathcal{T}_\mathfrak{p}, V)$: if $g \in C^0(\mathcal{T}_\mathfrak{p}, V)$,

$$(dg)(e) = g(t(e)) - g(s(e)).$$

It is easy to see that d is surjective: fix a $v_0 \in \mathcal{V}(\mathcal{T}_\mathfrak{p})$. Then for $f \in C^1(\mathcal{T}_\mathfrak{p}, V)$, define $g \in C^0(\mathcal{T}_\mathfrak{p}, V)$ by

$$g(v) = \sum_{e: v_0 \to v} f(e)$$

Then $dg = f$.

Thus we have a short exact sequence of Γ-modules:

$$0 \to V \to C^0(\mathcal{T}_\mathfrak{p}, V) \xrightarrow{d} C^1(\mathcal{T}_\mathfrak{p}, V) \to 0.$$

Put $C^0_{\mathrm{har}}(\mathcal{T}_\mathfrak{p}, V) = d^{-1}(C^1_{\mathrm{har}}(\mathcal{T}_\mathfrak{p}, V))$. Then

$$0 \to V \to C^0_{\mathrm{har}}(\mathcal{T}_\mathfrak{p}, V) \xrightarrow{d} C^1_{\mathrm{har}}(\mathcal{T}_\mathfrak{p}, V) \to 0.$$

This gives the associated long exact sequence

$$0 \to V^\Gamma \to C^0_{\mathrm{har}}(\mathcal{T}_\mathfrak{p}, V)^\Gamma \xrightarrow{d} C^1_{\mathrm{har}}(\mathcal{T}_\mathfrak{p}, V)^\Gamma \xrightarrow{\partial} H^1(\Gamma, V).$$

A direct computation shows that the connecting homomorphism ∂ is equal to $-\kappa^{\mathrm{sch}}_\Gamma$. Hence it suffices to prove that $V^\Gamma = C^0_{\mathrm{har}}(\mathcal{T}_\mathfrak{p}, V)^\Gamma$.

To prove this we need a rational structure on the $\mathbf{C}_p[\Gamma]$-module V. Let K be the Galois closure of F in $\overline{\mathbf{Q}}$. Then K is again totally real. Fix an embedding of K into $\mathbf{C}_p$. We are going to define a K-vector space U, with K-linear action of Γ, such that $U \otimes_K \mathbf{C}_p \cong V$ as Γ-module. Then $V^\Gamma = U^\Gamma \otimes_K \mathbf{C}_p$. While $C^0_{\mathrm{har}}(\mathcal{T}_\mathfrak{p}, V) \neq C^0_{\mathrm{har}}(\mathcal{T}_\mathfrak{p}, U) \otimes_K \mathbf{C}_p$, we do have $C^0_{\mathrm{har}}(\mathcal{T}_\mathfrak{p}, V)^\Gamma = (C^0_{\mathrm{har}}(\mathcal{T}_\mathfrak{p}, U) \otimes_K \mathbf{C}_p)^\Gamma$ since $\Gamma\backslash\mathcal{T}_\mathfrak{p}$ is finite. The latter is equal to $C^0_{\mathrm{har}}(\mathcal{T}_\mathfrak{p}, U)^\Gamma \otimes_K \mathbf{C}_p$. So it suffices to show that $U^\Gamma = C^0_{\mathrm{har}}(\mathcal{T}_\mathfrak{p}, U)^\Gamma$.

We now define such a K-structure for V. Say the quaternion algebra B is defined by

$$B = F + F\alpha + F\beta + F\alpha\beta$$

where $\alpha\beta = -\beta\alpha$, $\alpha^2 = a$, $\beta^2 = b$, with a, b totally negative elements of F. Now since we have fixed an embedding of K into $\mathbf{C}_p$, we can identify I, the set of embeddings of F into $\overline{\mathbf{Q}}_p$, with the set of embeddings of F into K. For each embedding $\sigma \in I$, put

$B_{K,\sigma} := B \otimes_{F,\sigma} K$. Let $B^0_{K,\sigma} := \{\xi \in B_{K,\sigma},\ \mathrm{Trd}_{B_{\sigma,K}/K}(\xi) = 0\}$ (spanned over K by $\alpha \otimes 1, \beta \otimes 1, \alpha\beta \otimes 1$ for instance), and by $(B_{K,\sigma})_1 = \{\xi \in B_{K,\sigma},\ \mathrm{Nrd}_{B_{\sigma,K}/K}(\xi) = 1\}$. Here $\mathrm{Trd}_{B_{K,\sigma}/K}, \mathrm{Nrd}_{B_{K,\sigma}/K}$ are the reduced trace and reduced norm on $B_{K,\sigma}$ (which are induced from that of B). We have the right action of $(B_{K,\sigma})_1$ on $B^0_{K,\sigma}$ by conjugation.

There is a K-valued positive definite inner product $(\ ,\ )$ on $B^0_{K,\sigma}$, defined as follows: for $\theta, \xi \in B^0_{K,\sigma}$,

$$(\theta, \xi) = \mathrm{Trd}_{B_{K,\sigma}/K}\big(\theta\xi^*\big)$$

where $\xi \mapsto \xi^*$ is the canonical involution of $B_{K,\sigma}$ (again induced from that of B). It is immediately seen to be invariant under $(B_{K,\sigma})_1$.

Define $W_{1,\sigma}$ to be the K-dual of $B^0_{K,\sigma}$. It inherits the $(B_{K,\sigma})_1$-invariant inner product from $B^0_{K,\sigma}$. For example, define $X_{1,\sigma}, X_{2,\sigma}, X_{3,\sigma}$ to be the elements of $W_{1,\sigma}$ dual to $\alpha \otimes 1, \beta \otimes 1, \alpha\beta \otimes 1$. Then

$$(X_{1,\sigma}, X_{1,\sigma}) = -1/a^\sigma, \qquad (X_{2,\sigma}, X_{2,\sigma}) = -1/b^\sigma, \qquad (X_{3,\sigma}, X_{3,\sigma}) = 1/(ab)^\sigma.$$

Let $(B_{K,\sigma})_1$ act on the left on $W_{1,\sigma}$, by dualizing the right action of $(B_{K,\sigma})_1$ on $B^0_{K,\sigma}$.

For integer $m \geq 1$, define $W_{m,\sigma}$ to be the m-th symmetric power of $W_{1,\sigma}$, with the induced action of $(B_{K,\sigma})_1$. The dimension of $W_{m,\sigma}$ over K is $\binom{m+2}{2}$, and can be viewed as the space of homogeneous polynomials in $X_{1,\sigma}, X_{2,\sigma}, X_{3,\sigma}$ of degree m with coefficients in K. It inherits the $(B_{K,\sigma})_1$-invariant inner product from $W_{1,\sigma}$.

Now denote by $\Delta_\sigma : W_{m,\sigma} \to W_{m-2,\sigma}$ the Laplacian:

$$\Delta_\sigma = -\frac{1}{a^\sigma}\frac{\partial^2}{\partial X^2_{1,\sigma}} - \frac{1}{b^\sigma}\frac{\partial^2}{\partial X^2_{2,\sigma}} + \frac{1}{(ab)^\sigma}\frac{\partial^2}{\partial X^2_{3,\sigma}}.$$

The map Δ_σ is surjective (with the convention that $W_{-1,\sigma} = 0$, and $W_{0,\sigma} = K$) and commutes with the action of $(B_{K,\sigma})_1$. Hence if we define $U_{m,\sigma} := \ker(\Delta_\sigma : W_{m,\sigma} \to W_{m-2,\sigma})$. Then $U_{m,\sigma}$ has dimension $2m+1$ over K, with the action of $(B_{K,\sigma})_1$. Furthermore when we extend scalars from K to $\mathbf{C}_p$, then $U_{m,\sigma} \otimes_K \mathbf{C}_p$ becomes isomorphic to $(\mathrm{Sym}^{2m}(\mathbf{C}_p^2))^*$, with the isomorphism commuting with the action of $(B_{K,\sigma})_1$ [9, Corollary 2.16, part (1)]. Here $(B_{K,\sigma})_1$ acts on $(\mathrm{Sym}^{2m}(\mathbf{C}_p^2))^*$ via (here $\mathfrak{q}|p$ is the prime for which $\sigma \in I_\mathfrak{q}$):

$$(B_{K,\sigma})_1 \to (B_{K,\sigma} \otimes_K \mathbf{C}_p)_1 = (B \otimes_{F,\sigma} \mathbf{C}_p)_1 = (B_\mathfrak{q} \otimes_{F_\mathfrak{q},\sigma} \mathbf{C}_p)_1 \overset{(2.1)}{\cong} \mathrm{SL}_2(\mathbf{C}_p). \quad \text{(A.1)}$$

In particular when restricted to B_1, this can be described as follows: for $\gamma \in B_1$, let $\begin{pmatrix} a^\sigma & b^\sigma \\ c^\sigma & d^\sigma \end{pmatrix} \in \mathrm{SL}_2(\mathbf{C}_p)$ be the image of $\gamma \otimes 1 \in (B_{K,\sigma})_1$ under the map (A.1). Then for $h \in (\mathrm{Sym}^{2m}(\mathbf{C}_p^2))^*$, the action of $\gamma \otimes 1$ sends h to h', where

$$h'\big(P(t)\big) = h\bigg(\big(c^\sigma t + d^\sigma\big)^{2m} \cdot P\bigg(\frac{a^\sigma t + b^\sigma}{c^\sigma t + d^\sigma}\bigg)\bigg)$$

for all $P(t) \in \mathrm{Sym}^{2m}(\mathbf{C}_p^2)$.

In particular, it follows that for $V = \bigotimes_{\sigma\in I} V_\sigma(n_\sigma, v_\sigma)$, with $n_\sigma = 2m_\sigma$, we can put $U = \bigotimes_{\sigma\in I} U_{m_\sigma,\sigma}$. The action of B_1 on U is via the tensor product action coming from the embedding $B_1 \to \prod_{\sigma\in I}(B_{K,\sigma})_1$. Then V is isomorphic to $U \otimes_K \mathbf{C}_p$, and the isomorphism respects the action of Γ. We also see that U inherits a Γ-invariant inner product $\langle\ ,\ \rangle$ from the inner products on each $U_{m_\sigma,\sigma}$.

The Γ-invariant K-valued inner product $\langle\ ,\ \rangle$ on U can be used to define non-degenerate bilinear forms on $C^0(\mathcal{T}_\mathfrak{p}, U)^\Gamma$ and $C^1(\mathcal{T}_\mathfrak{p}, U)^\Gamma$:

$$\langle g_1, g_2\rangle = \sum_{v\in\Gamma\backslash\mathcal{V}(\mathcal{T}_\mathfrak{p})} \big\langle g_1(v), g_2(v)\big\rangle \text{ on } C^0(\mathcal{T}_\mathfrak{p}, U)^\Gamma,$$

$$\langle f_1, f_2\rangle = \sum_{e\in\Gamma\backslash\mathcal{E}^0(\mathcal{T}_\mathfrak{p})/\{\pm1\}} \big\langle f_1(e), f_2(e)\big\rangle \text{ on } C^1(\mathcal{T}_\mathfrak{p}, U)^\Gamma.$$

Here $\mathcal{E}(\mathcal{T}_\mathfrak{p})/\{\pm1\}$ means we take the edges of $\mathcal{E}(\mathcal{T}_\mathfrak{p})$ modulo orientation. Since K is real, these are actually inner products, i.e. positive definite.

Define $\delta : C^1(\mathcal{T}_\mathfrak{p}, U) \to C^0(\mathcal{T}_\mathfrak{p}, U)$ by

$$\delta f(v) = \sum_{t(e)=v} f(e).$$

Then $f \in C^1_{\mathrm{har}}(\mathcal{T}_\mathfrak{p}, U)$ if and only if $\delta f \equiv 0$.

By a direct computation, we have for $f \in C^0(\mathcal{T}_\mathfrak{p}, U)^\Gamma, g \in C^1(\mathcal{T}_\mathfrak{p}, U)^\Gamma$:

$$\langle f, dg\rangle = \langle \delta f, g\rangle.$$

Now we can complete the proof as follows. Suppose that $g \in C^0_{\mathrm{har}}(\mathcal{T}_\mathfrak{p}, U)^\Gamma$. Then $dg \in C^1_{\mathrm{har}}(\mathcal{T}_\mathfrak{p}, U)^\Gamma$, so $\delta dg \equiv 0$. Thus we have

$$\langle dg, dg\rangle = \langle \delta dg, g\rangle = 0.$$

This implies $dg \equiv 0$ by the positive definiteness of $\langle\ ,\ \rangle$. Thus $g \in U^\Gamma$.

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