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Elliptic genus of E-strings

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Abstract: We study a family of 2d $\mathcal{N} = (0, 4)$ gauge theories which describes at low energy the dynamics of E-strings, the M2-branes suspended between a pair of M5 and M9 branes. The gauge theory is engineered using a duality with type IIA theory, leading to the D2-branes suspended between an NS5-brane and 8 D8-branes on an O8-plane. We compute the elliptic genus of this family of theories, and find agreement with the known results for single and two E-strings. The partition function can in principle be computed for arbitrary number of E-strings, and we compute them explicitly for low numbers. We test our predictions against the partially known results from topological strings, as well as from the instanton calculus of 5d Sp(1) gauge theory. Given the relation to topological strings, our computation provides the all genus partition function of the refined topological strings on the canonical bundle over $\frac{1}{2}K3$.

Keywords: Field Theories in Higher Dimensions, Field Theories in Lower Dimensions, M-Theory, Solitons Monopoles and Instantons

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1 Introduction

Six dimensional superconformal theories with (2,0) and (1,0) supersymmetry enjoy a special status among all superconformal theories: they are at the highest possible dimension. They play a key role in various aspects of string dualities as well as in obtaining lower dimensional supersymmetric systems upon compactification. They are rather enigmatic as they include tensionless self-dual strings as their building blocks.

The study of these theories has recently intensified, leading to computations of their superconformal indices [1–4], the elliptic genera of the self-dual strings in the Coulomb branch [5–7] (see [8] for an earlier work), as well as a partial classification of 6d superconformal theories [9–11]. The aim of this paper is to take a step forward in this direction, in particular focusing on one of the most basic $(1, 0)$ superconformal theories. The theory is known to arise in heterotic strings for small E_8 instantons [12–14], and also when an M5 brane approaches the M9 brane boundary [13, 14]. It also has an F-theory dual description given by blowing up a point on $\mathbb{C}^2$ base of F-theory [15–17]. This superconformal theory has an E_8 global symmetry. It also has a one dimensional Coulomb branch, parameterized by a real scalar in the (1,0) tensor multiplet. In the M-theory setup, the scalar parameterizes the distance between M5 and M9 branes [18]. In F-theory setup, it parameterizes the size of the $\mathbb{P}^1$ obtained by blowing up a point. On the Coulomb branch this theory has light strings, known as E-strings [19]. In the M-theory setup they arise by M2 branes stretched between M5 brane and M9 brane. In F-theory setup they arise by wrapping D3 branes on the blown up $\mathbb{P}^1$. It is natural to ask whether one can find a nice description of E-strings. The main aim of this paper is to find such a description and use it to compute the twisted partition function of such strings on T^2. More precisely we would be computing the elliptic genus of E-strings on T^2. Knowing the elliptic genus of E-strings is useful in its own right, as well as for uncovering aspects of the superconformal theory. For example, a basic quantity one may wish to compute for a superconformal theory is its superconformal index, which involves the computation of its partition function on $S^1 \times S^5$ with suitable fugacities turned on along S^1. As was argued in [2, 3] (see also [20, 21]), the computation of the superconformal index reduces to an integral over the Coulomb branch where the integrand consists of three copies of elliptic genus of the corresponding strings.

If one is computing supersymmetry protected quantities, such as elliptic genus, we can change parameters to make the computation easy. In particular one can change parameters and use string dualities to find a suitable description of the resulting strings. This strategy was employed in particular for M-strings and their orbifolds [5, 6]. Two basic ways were used to compute the elliptic genus of the M-strings: one was to use string dualities to map the 2d theory to a super-Yang-Mills type gauge theory and use the technique developed recently [22–24] to compute their elliptic genera. The other way was to use the relation of the elliptic genus to BPS quantities upon circle compactification of these theories, that can in principle be computed using topological strings.

In the context of E-strings we employ the former method, and identify the gauge theory which captures their low energy physics. This is done by considering the duality of M-theory with type IIA, by introducing a circle transverse to M5 brane, leading to a system involving NS5-brane and where the M9 brane is replaced by O8 plane with 8 D8 branes on it. The M2 branes suspended between M5 and M9 branes map to D2 branes suspended between NS5-brane and O8-D8 pair. We find a simple $(0, 4)$ supersymmetric quiver describing this system with $O(n)$ gauge symmetry, where n denotes the number of suspended M2 branes. We use it to compute the elliptic genus of n E-strings by employing the techniques developed in [23, 24].

The other method of computing the elliptic genus of E-string involves the F-theory picture. Namely, we compactify the theory on a circle leading to an M-theory description, and consider the BPS states of wrapped M2 branes, which correspond to E-strings wound around S^1 [25]. M-theory geometry involves the canonical bundle over $\frac{1}{2}K3$. As is well known, the BPS states of M2 branes wrapped on it, are captured by topological string amplitudes [26, 27]. In this context the (refined) topological string for $\frac{1}{2}K3$ has been computed to a high genus [28, 29], though an all genus answer is not available. So our method leads to a complete answer for refined topological string on $\frac{1}{2}K3$. Our answer can also be related to $\mathcal{N} = 4$ Yang-Mills in $d = 4$ in two different ways. In the F-theory setup, E-strings arise by wrapping D3 branes on a $\mathbb{P}^1$. From this perspective the elliptic genus of n E-strings gets mapped to the study of n D3 branes on $T^2 \times \mathbb{P}^1$, i.e. the partition function of $\mathcal{N} = 4$ U(n) Yang-Mills on this geometry. Except that the coupling constant of Yang-Mills τ is not a constant and varies over $\mathbb{P}^1$ according to the complex structure of the elliptic curve given by

$$y^2 = x^3 + f_4(z)x + g_6(z)$$

where z parameterizes the $\mathbb{P}^1$ and f_4 and g_6 are polynomials of degree 4 and 6 respectively. Note that this takes into account the S-duality of U(n) Yang-Mills. Moreover lifting this to M-theory leads to n M5 branes on $T^2 \times \frac{1}{2}K3$, which gets mapped to U(n) $\mathcal{N} = 4$ Yang-Mills on $\frac{1}{2}K3$ [30] (for the SU(2) case see [31] and for computations in related cases see [32]).

Explicit computations for the elliptic genus are now straightforward, but somewhat cumbersome. Nevertheless we carry it out explicitly for the case of n E-strings for $n = 1, 2, 3, 4$, and also explain the concrete procedures needed to compute the elliptic genus in the case with general n. The case with $n = 1$ was already known in [19], and the case with $n = 2$ was recently found in [7]. For the other two cases we check our results against partial results from topological strings on $\frac{1}{2}K3$ (where low genus answer is known). We also check them at $n = 4$ against a recent proposal of [33], where the elliptic genus was proposed at a special value of E_8 fugacities with reduced symmetry SO(8) × SO(8) ⊂ E_8. In all these cases we find agreements with our computations.

Finally, we explain an alternative method to compute the E-string elliptic genus, from the instanton calculus of 5d SYM theories with Sp(1) gauge group and 8 fundamental hypermultiplets. The index for k instantons captures the k'th order coefficient of the elliptic genus expanded in the modular parameter, but keeps the information on all higher E-strings' spectrum at this order. It was recently shown in [34] how to compute this index. Making double expansions of the indices of our 2d gauge theory and the instanton quantum mechanics, we confirm that the indices computed from the two approaches agree with each other.

The organization of this paper is as follows: in section 2 we describe the basic type IIA brane setup. In section 3 we use this to compute the elliptic genera of E-strings. We give the explicit details for 1, 2, 3, 4 E-strings and indicate how the higher case works. We also compare with (partial) known results. In section 4 we also formulate how the E-string partition function can be computed using 5 dimensional Yang-Mills instantons, and compare the results with those obtained in section 3. In section 5 we present some concluding remarks. Some technical details are relegated to the appendices.

2 The brane setup and the 2d (0, 4) gauge theories

We construct a brane system in the type IIA string theory, which at low energy engineers the 6d E_8 SCFT and the 2d CFT for E-strings. We first take an NS5-brane to wrap the 013456 directions, located at $x^2 = L$ (> 0), $x^7 = x^8 = x^9 = 0$. An O8-plane and 8 D8-branes (or 16 D8-branes in the covering space of orientifold) wrap 013456789 directions, located at $x^2 = 0$. To describe E-strings, n D2-branes are stretched between the NS5 and 8-brane system ($0 < x^2 < L$), occupying 012 directions. This brane system has SO(4) $\sim$ SU(2)$_L$ × SU(2)$_R$ and SO(3) $\sim$ SU(2)$_I$ symmetries which rotate 3456 and 789

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	0	1	2	3	4	5	6	7	8	9
NS5	•	•		•	•	•	•			
D8-O8	•	•		•	•	•	•	•	•	•
D2	•	•	•							

Table 1. Brane configuration for the E-strings.

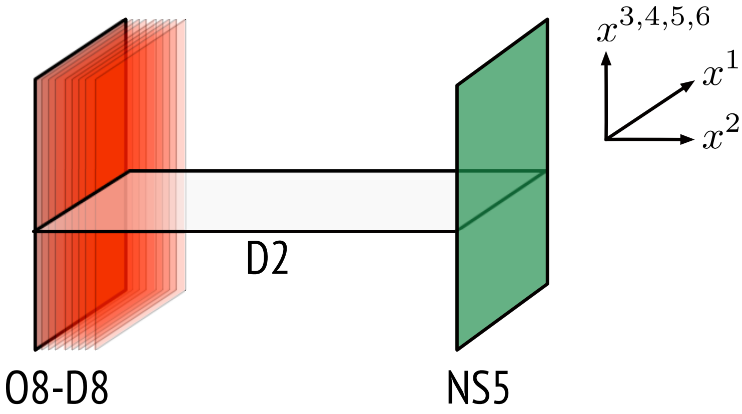

Figure 1. The type IIA brane configuration for the E-strings.

directions. We denote by $\alpha, \beta, \cdots = 1, 2$, $\dot{\alpha}, \dot{\beta}, \cdots = 1, 2$ and $A, B, \cdots = 1, 2$ the doublet indices of these three SU(2) symmetries. See table 1 and figure 1.

The M-theory uplift of this brane configuration, with extra circle direction labeled by x^{10}, is given as follows. The NS5-brane lifts to the M5-brane transverse to the x^{10} direction. The D8-O8 system uplifts to an M9-plane, or the Horava-Witten wall [18], longitudinal in x^{10} direction. In order to get a weakly-coupled type IIA string theory at low energy, one has to turn on suitable E_8 Wilson line along x^{10} to break $E_8 \to \mathrm{SO}(16)$ [13]. See our section 4 for more details. D2-branes uplift to M2-branes transverse in x^{10}. In the strong coupling limit of the type IIA theory, the radius of the M-theory circle becomes large. The geometry $\mathbb{R}^3 \times S^1$ transverse to the 5-brane is replaced by $\mathbb{R}^4$. So the brane configuration contains the M5-M9 system, in the Coulomb branch of the 6d E_8 CFT. M2-branes suspended between them are the E-strings.

At an energy scale much lower than L^{-1}, one obtains a 2d QFT living at the intersection of these branes. At $g_{\mathrm{YM}} \ll E \ll L^{-1}$ with $g^2_{\mathrm{YM}} \sim \frac{g_s}{L\ell_s}$, where ℓ_s, g_s are the string scale and the coupling constant, one obtains a weakly coupled 2d Yang-Mills description with coupling constant g_{YM}. (One can take g_s to be sufficiently small, and L to be sufficiently larger than ℓ_s.) When $E \ll g_{\mathrm{YM}}$, the 2d Yang-Mills theory is strongly coupled and is expected to flow to an interacting SCFT. In terms of the Planck scale $\ell_P \sim g_s^{1/3}\ell_s$ of M-theory and the radius $R \sim g_s \ell_s$ of the x^{10} circle, the strong coupling regime of the 2d Yang-Mills theory is $E \ll \frac{R}{L^{1/2}\ell_P^{3/2}}$. L is related to the VEV v of the scalar in the 6d tensor

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multiplet by $L \sim v\ell_P^3$. So the low energy limit is $E \ll \frac{R}{v^{1/2}\ell_P^3}$. In the Coulomb branch with fixed v, this low energy limit of the 2d theory is obtained by taking the M-theory limit $R \to \infty$, in which case the system describes E-strings as explained in the previous paragraph. Thus our 2d gauge theory describes E-strings at its strong coupling fixed point.

Let us comment on the enhanced IR symmetries. We first consider the $SO(3) \times U(1)$ acting on $\mathbb{R}^3 \times S^1$. In the M-theory limit, this enhances to $SO(4) \sim SU(2)_l \times SU(2)_r$ of $\mathbb{R}^4$. $SO(3) \sim SU(2)_I$ is identified as the diagonal combination of $SU(2)_r$ and $SU(2)_l$. On the other hand, from the viewpoint of 6d superconformal symmetry, $SU(2)_r$ is the R-symmetry of the 6d (1,0) SCFT and $SU(2)_l$ is a flavor symmetry. So it might appear that our 2d gauge theory is probing only a combination of the R-symmetry and a flavor symmetry. However, in the rank 1 system with only one M5-brane, the extra flavor $SU(2)_l$ completely decouples with the 6d CFT. For instance, these can be seen by studying the instanton partition functions of circle reduced 5d SYM [34], which will also be the subject of our section 4. Thus we can identify SO(3) visible in our 2d UV theory as the superconformal R-symmetry of the 6d CFT. E-strings of the higher rank 6d SCFTs which see $SU(2)_l$ are discussed in [35, 36].

We also discuss the E_8 global symmetry. The 2d UV theory exhibits SO(16) symmetry only. This should enhance to E_8 in the IR, which is naturally expected from the brane perspective. Namely, the type IIA brane system is obtained by compactifying M-theory brane system with an E_8 Wilson line which breaks E_8 to SO(16). The IR limit on the 2d gauge theory is the strong coupling limit, which is the decompactification limit of the M-theory circle. So in this limit, the information on the Wilson line will be invisible, making us to expect an IR E_8 enhancement. In section 3, we shall compute the elliptic genera of these gauge theories at various values of n, which will be invariant under the E_8 Weyl symmetry and support the E_8 enhancement.

Let us study the SUSY of this system. The D2, D8 SUSY are associated with the projectors Γ^{012} and $\Gamma^{013456789}\Gamma^{11} \sim \Gamma^2$ respectively, while the NS5-brane projector is $\Gamma^{01}\Gamma^{3456}$. Various combinations of branes share different SUSY. We list the following projectors which should assume definite eigenvalues for the type IIA SUSY parameter ϵ, for various combinations of branes:

$$\text{D2-D8-NS5}: \ \Gamma^{01}\,, \quad \Gamma^2\,, \quad \Gamma^{3456} \tag{2.1}$$

$$\text{D2-NS5}: \ \Gamma^{01}\Gamma^2\,, \quad \Gamma^{01}\Gamma^{3456} \tag{2.2}$$

$$\text{D2-D8-O8}: \ \Gamma^{01}\,, \quad \Gamma^2\,. \tag{2.3}$$

The projectors (2.1) will yield the SUSY preserved by our system. The SUSY given by (2.2) and (2.3) will constrain the boundary conditions of the 3d D2-brane fields at the two ends of the segment along x^2. Let us investigate them in more detail. The type IIA supercharges with 32 components can be arranged to be eigenstates of $\Gamma^{01}, \Gamma^{3456}, \Gamma^2$. The eigenspinors of Γ^{01} are 2d chiral spinors, while those of Γ^{3456} belong to either $(\mathbf{2},\mathbf{1})$ or $(\mathbf{1},\mathbf{2})$ representations of $SU(2)_L \times SU(2)_R$. The 32 supercharges decompose into the sum of the $(\mathbf{2},\mathbf{1},\mathbf{2})_{\pm\pm} \oplus (\mathbf{1},\mathbf{2},\mathbf{2})_{\pm\pm}$ representations of $SU(2)_L \times SU(2)_R \times SU(2)_I$ with all four possible choices of $\pm\pm$, where the first/second $\pm$ subscripts denote 2d chirality and Γ^2 eigenvalues, respectively. The SUSY preserved by various combinations of branes are given by

$$\text{D2-D8-NS5}: \ (\mathbf{1},\mathbf{2},\mathbf{2})_{-+} \tag{2.4}$$
$$\text{D2-NS5}: \ (\mathbf{2},\mathbf{1},\mathbf{2})_{+-} \oplus (\mathbf{1},\mathbf{2},\mathbf{2})_{-+} \tag{2.5}$$
$$\text{D2-D8-O8}: \ (\mathbf{2},\mathbf{1},\mathbf{2})_{-+} \oplus (\mathbf{1},\mathbf{2},\mathbf{2})_{-+}\,. \tag{2.6}$$

(2.4) yields the 2d $(0,4)$ SUSY, which we write as $Q_-^{\dot\alpha A}$. (2.5) yields 2d $(4,4)$ SUSY $Q_+^{\alpha A}$, $Q_-^{\dot\alpha A}$. (2.6) yields 2d $(0,8)$ SUSY $Q_-^{\alpha A}$, $Q_-^{\dot\alpha A}$. $\pm$ subscripts of Q denote 2d left/right chiral spinors.

We study the field contents of the 2d $\mathcal{N}=(0,4)$ gauge theory. This is obtained by starting from the 3d field theory living on D2-branes, together with the boundary degrees of freedom at $x^2=0,L$, and then taking a 2d limit when $E\ll L^{-1}$. The 3d fields living in the region $0<x^2<L$ are

$$\begin{aligned}\text{D2-D2}: A_\mu \ \ (\mu=0,1,2)\ ; \ \ X^I\sim\varphi^{\alpha\dot\beta} \ \ (I=3,4,5,6)\ ; \ \ X^{I'} \ \ (I'=7,8,9)\\ \lambda \ \ (\text{has 16 components, satisfying } \ \Gamma^{11}\lambda=-\lambda)\,.\end{aligned} \tag{2.7}$$

The D2-D2 fields are in adjoint representation of U(n). One also finds boundary degrees at the brane intersections. At the intersection of D2-D8, open strings provide 2d Fermi multiplet fields which we write as Ψ_l $(l=1,\cdots,16)$. They will be in the bi-fundamental representation of $O(n)\times$SO(16) (after introducing the $O8^-$ orientifold). Ψ_l are left-moving Majorana-Weyl spinors. The maximal supersymmetry on D2-brane worldvolume is parameterized by $\frac{1+\Gamma^{11}}{2}\epsilon$, where ϵ is an eigenvector of Γ^{012} (and further projection conditions listed above at the boundaries).

Let us consider the boundary conditions of the 3d fields. At the two ends $x^2=0,L$, we shall find separate boundary conditions. As our goal is to obtain the 2d theory, we shall only keep the zero modes of the 3d fields along the x^2 direction. This means that we shall keep the bosonic fields satisfying the Neumann boundary conditions on both ends, and the fermionic fields which survive suitable projection conditions at both ends. The SUSY conditions for the D2-D2 fields at $x^2=0,L$ take the form of

$$(x^2\text{ component of supercurrent})\sim\text{tr}\left(\bar\epsilon(1+\Gamma^{11})\Gamma^{MN}F_{MN}\Gamma_2\lambda\right)=0 \tag{2.8}$$

in the 10d notation with $M,N=0,\cdots,9$. ϵ is chosen to be $(4,4)$ on D2-NS5 $(x^2=L)$, and $(0,8)$ on D2-D8 $(x^2=0)$. One can follow the strategy of [37] to obtain the SUSY boundary conditions. With given SUSY ϵ, one first imposes suitable bosonic boundary condition, depending on which branes D2's are ending on. Then the condition (2.8) would determine the boundary condition for the fermions λ.

We study the D2-NS5 boundary condition first, for which $\bar\epsilon$ is taken to be $(\mathbf{2},\mathbf{1},\mathbf{2})_{+-}\oplus(\mathbf{1},\mathbf{2},\mathbf{2})_{-+}$. The D2-D2 fermion λ satisfies $\lambda=-\Gamma^{11}\lambda$, where $\Gamma^{11}\sim\Gamma^{01}\Gamma^{3456}\Gamma^{78}\Gamma^{29}$. So depending on the eigenvalues of Γ^{01}, Γ^{3456}, Γ^{78} (the spin of SU$(2)_I$), λ can be decomposed into

$$(\text{SU}(2)_L,\text{SU}(2)_R,\text{SU}(2)_I)_{\Gamma^{01}}=(\mathbf{2},\mathbf{1},\mathbf{2})_+\oplus(\mathbf{2},\mathbf{1},\mathbf{2})_-\oplus(\mathbf{1},\mathbf{2},\mathbf{2})_+\oplus(\mathbf{1},\mathbf{2},\mathbf{2})_-\,, \tag{2.9}$$

and Γ^{29} eigenvalues are determined from $\Gamma^{11}\lambda = -\lambda$. Unlike ϵ, the Γ^2 eigenvalue cannot be specified for λ, since it does not commute with Γ^{29}. We start from the boundary conditions for the bosonic fields that we know for D2-NS5:

$$F_{\mu 2} = 0\,, \quad D_2 X^I = 0\,, \quad X^{I'} = 0 \tag{2.10}$$

with $\mu = 0, 1$, $I = 3, 4, 5, 6$, $I' = 7, 8, 9$. This provides the following constraints on λ:

$$0 = \bar{\epsilon}\lambda = \bar{\epsilon}\Gamma^{\mu 2 I}\lambda = \bar{\epsilon}\Gamma^{IJ}\Gamma^2\lambda = \bar{\epsilon}\Gamma^{I'}\lambda\,. \tag{2.11}$$

This requires λ to be in

$$(\mathrm{SU}(2)_L, \mathrm{SU}(2)_R, \mathrm{SU}(2)_I)_{\Gamma^{01}} = (\mathbf{2}, \mathbf{1}, \mathbf{2})_- \oplus (\mathbf{1}, \mathbf{2}, \mathbf{2})_+\,, \tag{2.12}$$

namely, with a right mover $\lambda_-^{\alpha A}$ and a left mover $\lambda_+^{\dot{\alpha} A}$. (The former will belong to a 2d $(0, 4)$ hypermultiplet and the latter will belong to a 2d $(0, 4)$ vector multiplet.)

Now we consider the D2-D8-O8 boundary conditions. The effect of having 8 D8-branes is simply adding Fermi multiplet fields as explained above. So we focus on the effect of the O8-plane. Following [37], we consider the covering space of $x^2 > 0$ and consider the 3d SYM on $\mathbb{R}^{2,1}$. The reflection $x^2 \to -x^2$ of space is accompanied by an outer automorphism τ acting on $G = \mathrm{U}(n)$ gauge group. The algebra g of G decomposes into $g^{(+)} \oplus g^{(-)}$, where τ acts on $g^{(\pm)}$ as ± 1. In our case, $g^{(+)}$ is the algebra of $O(n) \subset \mathrm{U}(n)$, and $g^{(-)}$ forms a rank 2 symmetric representation of $O(n)$. So any adjoint-valued field Φ can be written as $\Phi = \Phi^{(+)} + \Phi^{(-)}$. The reflection is further accompanied by $X^I \to -X^I$ for $I = 3, \cdots, 9$. This is because odd number of scalars should flip sign for the net reflection to preserve the orientation of $\mathbb{R}^{9,1}$, e.g. to preserve Γ^{11} projection conditions in the 3d maximal SYM. Since the D2-D8-O8 boundary condition preserves SO(7) which rotates $I = 3, \cdots, 9$, all X^I's should be flipped. So the fields are required to be invariant under the net reflection:

$$A_\mu(x^2) = A_\mu^\tau(-x^2),\;\; A_2(x^2) = -A_2^\tau(-x^2),\;\; X_I(x^2) = -X_I^\tau(-x^2) \tag{2.13}$$

where $\Phi^\tau = \tau\Phi\tau^{-1}$, $\mu = 0, 1$ and $I = 3, \cdots, 9$. So at the fixed plane $x^2 = 0$, the boundary condition is given by

$$F_{\mu 2}^{(+)} = 0\,, \quad F_{\mu\nu}^{(-)} = 0\,, \quad D_2 X_I^{(-)} = 0\,, \quad X_I^{(+)} = 0 \quad (I = 3, \cdots, 9)\,. \tag{2.14}$$

$A_2(x^2)$ can be gauged away using x^2 dependent gauge transformation along the interval. We can again find the fermionic boundary conditions from (2.8). This requires

$$0 = \bar{\epsilon}\lambda^{(+)} = \bar{\epsilon}\Gamma^I\lambda^{(+)} = \bar{\epsilon}\Gamma^{IJ2}\lambda^{(+)}\,, \quad 0 = \bar{\epsilon}\Gamma^\mu\lambda^{(-)} = \bar{\epsilon}\Gamma^{\mu I 2}\lambda^{(-)} \tag{2.15}$$

with $\mu = 0, 1$ and $I, J = 3, \cdots, 9$. $\bar{\epsilon}$ is chosen to be (2.6). Solving these constraints, the $O(n)$ adjoint fermion $\lambda^{(+)}$ and the $O(n)$ symmetric fermion $\lambda^{(-)}$ are required to be in

$$\begin{aligned} \lambda^{(+)} &: (\mathrm{SU}(2)_L, \mathrm{SU}(2)_R, \mathrm{SU}(2)_I)_{\Gamma^{01}} = (\mathbf{2}, \mathbf{1}, \mathbf{2})_+ \oplus (\mathbf{1}, \mathbf{2}, \mathbf{2})_+ \\ \lambda^{(-)} &: (\mathrm{SU}(2)_L, \mathrm{SU}(2)_R, \mathrm{SU}(2)_I)_{\Gamma^{01}} = (\mathbf{2}, \mathbf{1}, \mathbf{2})_- \oplus (\mathbf{1}, \mathbf{2}, \mathbf{2})_-\,. \end{aligned} \tag{2.16}$$

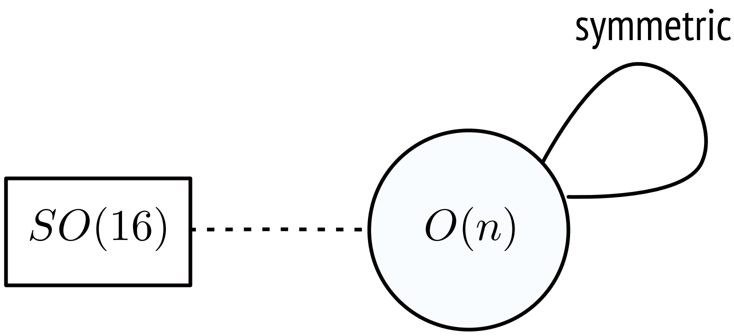

Figure 2. The quiver diagram of the 2d $\mathcal{N} = (0, 4)$ gauge theory for E-strings: solid/dotted lines denote hyper/Fermi multiplets, respectively.

We combine the D2-NS5 and D2-O8 boundary conditions to read off the 2d field contents. For bosons, requiring (2.10) and (2.14) yields the following 2d fields:

$$A_\mu^{(+)}\,, \quad X_I^{(-)} \sim \varphi_{\alpha\dot\beta} \quad (I = 3,4,5,6)\,. \tag{2.17}$$

For fermions, requiring (2.12) and (2.16) together, one finds that $\lambda_-^{\alpha A} \sim (\mathbf{2},\mathbf{1},\mathbf{2})_-$ is in the symmetric representation of $O(n)$, while $\lambda_+^{\dot\alpha A} \sim (\mathbf{1},\mathbf{2},\mathbf{2})_+$ is in the adjoint (i.e. antisymmetric) representation. So from the D2-D2 modes, we obtain the $(0,4)$ vector multiplet A_μ, $\lambda_+^{\dot\alpha A}$ of $O(n)$, and also a $(0,4)$ hypermultiplet $\varphi_{\alpha\dot\beta}$, $\lambda_-^{\alpha A}$ in the symmetric representation of $O(n)$. So to summarize, one obtains the following 2d $\mathcal{N} = (0,4)$ field contents:

$$\begin{aligned} \text{vector}: &\quad O(n) \text{ antisymmetric} \quad (A_\mu,\ \lambda_+^{\dot\alpha A}) \\ \text{hyper}: &\quad O(n) \text{ symmetric} \quad (\varphi_{\alpha\dot\beta},\ \lambda_-^{\alpha A}) \\ \text{Fermi}: &\quad O(n)\times \mathrm{SO}(16) \text{ bifundamental} \quad \Psi_l\,. \end{aligned} \tag{2.18}$$

Figure 2 shows the quiver diagram of this gauge theory. One can check the $\mathrm{SO}(n)$ gauge anomaly cancelation of this chiral matter content. Note that we have no twisted hypermultiplets, whose scalars form doublets of $\mathrm{SU}(2)_I$ and fermions form doublets of $\mathrm{SU}(2)_R$.

We also explain how to get the full Lagrangian of this system. Viewing this as a special case of $\mathcal{N} = (0,2)$ supersymmetric system, it suffices to determine the two holomorphic functions $E_\Psi(\Phi_i)$, $J^\Psi(\Phi_i)$ for each Fermi multiplet Ψ, depending on the $(0,2)$ chiral multiplet fields Φ_i. We choose $Q \equiv Q_1^{\dot 1}$ and $Q^\dagger$ as the $(0,2)$ subset. To have $(0,4)$ SUSY, the E, J functions for the adjoint $(0,2)$ Fermi multiplet $\Theta \equiv (\lambda_+^{\dot 12}, \lambda_+^{\dot 21})$ in the $(0,4)$ vector multiplet are required to be [38]

$$J_\Theta = \varphi\tilde\varphi - \tilde\varphi\varphi\,, \quad E_\Theta = 0\,, \tag{2.19}$$

where $\varphi \equiv \varphi_{1\dot 1}$, $\tilde\varphi \equiv \varphi_{2\dot 1}$ are $(0,2)$ chiral multiplet scalars which transform under $Q \equiv Q_1^{\dot 1}$. Note that, if the $(0,4)$ theory has both hypermultiplets and twisted hypermultiplets, the full interaction has to be more complicated [38]. Without twisted hypermultiplets in our system, (2.19) provides the full interactions associated with Θ. This induces a bosonic

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potential of the form $|J_\Theta|^2$, as well as the Yukawa interaction. Extra Fermi multiplets in the $(0,2)$ viewpoint are Ψ_l from D2-D8-O8 modes, so we should also determine their E, J. E_{Ψ_l}, J^{Ψ_l} are simply zero, from SO(16) symmetry. With all the E, J functions determined, the supersymmetric action can be written down if $E^a J_a = 0$, where the index a runs over all $(0,2)$ Fermi multiplets. This condition is clearly met. With these data, the full action can be written down in a standard manner: see, for instance, [38, 39]. In our case, the bosonic potential consists of $|J_\Theta|^2$ and the usual D-term potential, making the D-term potential from the 'SU$(2)_R$ triplet' of D-terms. The classical Higgs branch moduli space, given by nonzero $\varphi, \tilde{\varphi}$, is real $4n$ dimensional. Semi-classically, these are the positions of n E-strings.

One can also compute the central charges of the IR CFT from our UV gauge theory. Once we know the correct superconformal R-symmetry of the IR SCFT, the (right-moving) central charge of the IR CFT can be computed in UV by the anomaly of the superconformal R-symmetry. We closely follow [38–40], which use the $(0,2)$ superconformal R-symmetry to determine the central charges.

In our $(0,4)$ system, a semi-classical description is allowed when $\varphi^{\alpha\dot{\beta}}$ scalars are large. This is the CFT associated with the classical Higgs branch [41]. In this CFT, the superconformal R-symmetry can only come from SU$(2)_I$ in the UV theory. This is because the right sector contains the $O(n)$ symmetric scalar $\varphi_{\alpha\dot{\beta}}$, and the superconformal R-symmetry should not act on it [41]. Following [38], let us choose the supercharge $Q \equiv Q^{\dot{1}2}$ and use the $(0,2)$ superconformal symmetry to determine the central charge. The right-moving central charge c_R is given by

$$c_R = 3\text{Tr}(\gamma^3 R^2)\,, \tag{2.20}$$

with $\gamma^3 = \pm 1$ for the right/left moving fermions, respectively, and the trace acquires an extra $\frac{1}{2}$ factor for real fermions. The $(0,2)$ R-charge R is normalized so that $R[Q] = -1$. In the Higgs branch CFT, this should be proportional to the Cartan of SU$(2)_I$, so we set $R = 2J_I$. Collecting the contribution from $O(n)$ symmetric $\lambda^{\alpha A}$ in the right sector and adjoint $\lambda^{\dot{\alpha} A}$ in the left sector, one obtains

$$c_R = 3 \times \frac{1}{2} \times \frac{n^2+n}{2} \times (4 \times 1^2) - 3 \times \frac{1}{2} \times \frac{n(n-1)}{2} \times (4 \times 1^2) = 6n\,. \tag{2.21}$$

The left moving central charge c_L is determined from c_R by the gravitational anomaly [39]:

$$c_R - c_L = \text{Tr}(\gamma^3) = \frac{1}{2} \times 4\frac{n^2+n}{2} - \frac{1}{2} \times 4\frac{n^2-n}{2} - \frac{1}{2} \times 16n = -6n \;\rightarrow\; c_L = 12n\,. \tag{2.22}$$

$c_L = 12n$ is consistent with the result obtained in [30] (where $c_L = 12n - 4$ was found after eliminating 4 from the decoupled center-of-mass degrees of freedom.) One can semiclassically understand some of these results, by studying the region with large value of the Higgs scalar $\varphi^{\alpha\dot{\beta}}$. $c_R = 6n$ comes from the n pairs of 4 scalars and 4 fermions for n E-strings. As for $c_L = 12n$, the $4n$ scalars in the left moving sector accounts for $4n$, and the $16n$ real fermions Ψ_l accounts for $8n$. For $n = 1$, we know that the last 8 is given by the $G = E_8$ current algebra at level $k = 1$ (with dual Coxeter number $c_2 = 30$) [13, 19], whose central charge is indeed $\frac{k|G|}{k+c_2} = \frac{248}{1+30} = 8$.

3 E-string elliptic genera from 2d gauge theories

We consider the elliptic genus of the 2d $(0,4)$ $O(n)$ gauge theory, constructed in the previous section. We pick the same $(0,2)$ SUSY as before, and define the elliptic genus as follows:

$$Z_n(q,\epsilon_{1,2},m_l)=\text{Tr}_{\text{RR}}\left[(-1)^F q^{H_L}\bar{q}^{H_R}e^{2\pi i\epsilon_1(J_1+J_I)}e^{2\pi i\epsilon_2(J_2+J_I)}\prod_{l=1}^{8}e^{2\pi i m_l F_l}\right]. \tag{3.1}$$

J_1, J_2 are the Cartans of $\text{SO}(4)\sim\text{SU}(2)_L\times\text{SU}(2)_R$ which rotate the 34 and 56 orthogonal 2-planes, and J_I is the Cartan of $\text{SU}(2)_I$. F_l are the Cartans of SO(16), which we expect to be the Cartans of enhanced E_8 in IR. Note that $H_R\sim\{Q,Q^\dagger\}$ with $Q=Q^{\dot{1}}_1$ and $Q^\dagger=-Q^{\dot{2}}_2$, and the remaining factors inside the trace commute with $Q,Q^\dagger$. Note also that, the 2d gauge theory itself has a noncompact Higgs branch spanned by $\varphi^{\alpha\dot{\beta}}$. They are given nonzero masses by turning on ϵ_1,ϵ_2, so that the path integral for this index does not have any noncompact zero modes. The interpretation of the zero modes from $\varphi^{\alpha\dot{\beta}}$ at $\epsilon_1,\epsilon_2=0$ is clearly the multi-particle positions, so by keeping nonzero $\epsilon_{1,2}$ we are computing the multi-particle index, as usual. The single particle spectrum can be extracted from the multi-particle index.

The index (3.1) for $\mathcal{N}=(0,2)$ gauge theories was studied in [23, 24], by computing the path integral of the gauge theory on T^2. There appear compact zero modes from the path integral, coming from the flat connections on T^2. [23, 24] first fix the flat connections, integrate over the nonzero modes, and then integrate (or sum) over the flat connections to obtain their final expression for the index.

Let us first explain the possible flat connections of our $O(n)$ gauge theories on T^2. These are given by two commuting $O(n)$ group elements U_1, U_2, the Wilson lines along the temporal and spatial circles of T^2. Note that $O(n)$ is a disconnected group so that U_1 and U_2 can each have two disconnected sectors, depending on whether their determinants are 1 or -1. The general $O(n)$ holonomies on T^2, up to conjugation, can be derived using a D-brane picture [42].[1] The $O(n)$ flat connections are the zero energy configurations of the n D2-branes and an O2-plane wrapping T^2. By T-dualizing twice along the torus, one obtains n D0-branes moving along the $T^2/\mathbb{Z}_2$ orientifold. The flat connections T-dualize to the positions of D0-branes on $T^2/\mathbb{Z}_2$. There are four O0-plane fixed points on the covering space T^2. It suffices for us to classify all possible positions of D0-branes. When two D0-branes on the covering space are paired as $\mathbb{Z}_2$ images of each other, they have one complex parameter u as their position. Some D0-branes can also be stuck at the $\mathbb{Z}_2$ fixed points without a pair: they are fractional branes on $T^2/\mathbb{Z}_2$, whose positions are freezed at the fixed points. So the classification of $O(n)$ flat connections reduces to classifying the possible fractional brane configurations.

When $n=2p$ is even, one can first have all $2p$ D0-branes to make p pairs. In this branch, one finds p complex moduli u_i $(i=1,\cdots,p)$. Another possibility is to form $p-1$ pairs to freely move, while having 2 fractional D-branes stuck at two of the 4 fixed points.

[1] If the gauge group is not $O(n)$ but, say Spin(n) as in [42], one has to make a variation of the simple D-brane argument that we shall present here.

Note that the two fractional branes have to be stuck at different fixed points: otherwise they can pair and leave the fixed point, being a special case of the first branch. There are 6 ways of choosing 2 fixed points among 4, so we obtain 6 more sectors. Finally, one finds a sector in which $p-2$ pairs freely move, while 4 fractional D-branes are stuck at 4 different fixed points (when $p \geq 2$). After T-dualizing, U_1, U_2 are exponentials of the D0-brane positions. The above 8 sectors are summarized by the following pairs of Wilson lines U_1, U_2, for $O(2p)$ with $p \geq 2$:

$$
\begin{aligned}
(\text{ee}):\ & U_1 = \text{diag}(e^{iu_{1i}\sigma_2})_p\,, && U_2 = \text{diag}(e^{iu_{2i}\sigma_2})_p\ ;\\
& U_1 = \text{diag}(e^{iu_{1i}\sigma_2}, 1, -1, -1, 1)_{p-2}\,, && U_2 = \text{diag}(e^{iu_{2i}\sigma_2}, 1, 1, -1, -1)_{p-2};\\
(\text{eo}):\ & U_1 = \text{diag}(e^{iu_{1i}\sigma_2}, 1, 1)_{p-1}\,, && U_2 = \text{diag}(e^{iu_{2i}\sigma_2}, 1, -1)_{p-1}\ ;\\
& U_1 = \text{diag}(e^{iu_{1i}\sigma_2}, -1, -1)_{p-1}\,, && U_2 = \text{diag}(e^{iu_{2i}\sigma_2}, 1, -1)_{p-1};\\
(\text{oe}):\ & U_1 = \text{diag}(e^{iu_{1i}\sigma_2}, 1, -1)_{p-1}\,, && U_2 = \text{diag}(e^{iu_{2i}\sigma_2}, 1, 1)_{p-1}\ ;\\
& U_1 = \text{diag}(e^{iu_{1i}\sigma_2}, 1, -1)_{p-1}\,, && U_2 = \text{diag}(e^{iu_{2i}\sigma_2}, -1, -1)_{p-1};\\
(\text{oo}):\ & U_1 = \text{diag}(e^{iu_{1i}\sigma_2}, 1, -1)_{p-1}\,, && U_2 = \text{diag}(e^{iu_{2i}\sigma_2}, 1, -1)_{p-1}\ ;\\
& U_1 = \text{diag}(e^{iu_{1i}\sigma_2}, 1, -1)_{p-1}\,, && U_2 = \text{diag}(e^{iu_{2i}\sigma_2}, -1, 1)_{p-1}\,.
\end{aligned}
\tag{3.2}
$$

(ee), (eo), (oe), (oo) are for U_1, U_2 in the even or odd elements of $O(n)$. The symbol 'diag' denotes a block-diagonalized matrix. The subscripts are the number of independent complex parameters. The parameters live on $u_i = u_{1i} + \tau u_{2i} \in \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$, where τ is related to our fugacity q by $q = e^{2\pi i \tau}$. For odd $n = 2p+1$ with $n \geq 3$, one can make a similar analysis. There are 4 cases in which one has 1 fractional brane stuck at one of the 4 fixed points, and 4 more cases (when $p \geq 1$) in which 3 fractional branes are stuck at three of the 4 fixed points. So one obtains the following 8 sectors, for $p \geq 1$:

$$
\begin{aligned}
(\text{ee}):\ & U_1 = \text{diag}(e^{iu_{1i}\sigma_2}, 1)_p\,, && U_2 = \text{diag}(e^{iu_{2i}\sigma_2}, 1)_p\ ;\\
& U_1 = \text{diag}(e^{iu_{1i}\sigma_2}, -1, -1, 1)_{p-1}\,, && U_2 = \text{diag}(e^{iu_{2i}\sigma_2}, 1, -1, -1)_{p-1};\\
(\text{eo}):\ & U_1 = \text{diag}(e^{iu_{1i}\sigma_2}, 1)_p\,, && U_2 = \text{diag}(e^{iu_{2i}\sigma_2}, -1)_p\ ;\\
& U_1 = \text{diag}(e^{iu_{1i}\sigma_2}, -1, -1, 1)_{p-1}\,, && U_2 = \text{diag}(e^{iu_{2i}\sigma_2}, 1, -1, 1)_{p-1};\\
(\text{oe}):\ & U_1 = \text{diag}(e^{iu_{1i}\sigma_2}, -1)_p\,, && U_2 = \text{diag}(e^{iu_{2i}\sigma_2}, 1)_p\ ;\\
& U_1 = \text{diag}(e^{iu_{1i}\sigma_2}, 1, -1, 1)_{p-1}\,, && U_2 = \text{diag}(e^{iu_{2i}\sigma_2}, -1, -1, 1)_{p-1};\\
(\text{oo}):\ & U_1 = \text{diag}(e^{iu_{1i}\sigma_2}, -1)_p\,, && U_2 = \text{diag}(e^{iu_{2i}\sigma_2}, -1)_p\ ;\\
& U_1 = \text{diag}(e^{iu_{1i}\sigma_2}, 1, 1, -1)_{p-1}\,, && U_2 = \text{diag}(e^{iu_{2i}\sigma_2}, 1, -1, 1)_{p-1}\,.
\end{aligned}
\tag{3.3}
$$

There are two exceptional cases. For $O(1)$, the four sectors in (3.3) with rank $p-1$ are absent. So we only have four rank 0 sectors

$$(U_1, U_2) = (1,1),\ (1,-1),\ (-1,1),\ (-1,-1)\,. \tag{3.4}$$

For $O(2)$, the second sector in (3.2) with rank $p-2$ is absent. So we have seven sectors

$$(U_1, U_2) = (e^{iu_1\sigma_2}, e^{iu_2\sigma_2}),\ (1,\sigma_3),\ (-1,\sigma_3),\ (\sigma_3,1),\ (\sigma_3,-1),\ (\sigma_3,\sigma_3),\ (\sigma_3,-\sigma_3)\,. \tag{3.5}$$

The Wilson lines can be more conveniently labeled by their exponents, which we call $u = (u_1, \cdots, u_n)$ for $O(n)$. In the 2×2 blocks $e^{iu_{1i}\sigma_2}, e^{iu_{2i}\sigma_2}$ with continuous elements, the associated two u parameters are given by the two eigenvalues $\pm(u_{1i} + \tau u_{2i})$. In the blocks with discrete numbers, we assign $u_i = 0$ for an eigenvalue pair $(1,1)$ of U_1, U_2, $u_i = \frac{1}{2}$ for an eigenvalue pair $(-1,1)$, $u_i = \frac{\tau}{2}$ for $(1,-1)$, and $u_i = \frac{1+\tau}{2}$ for $(-1,-1)$. For the above 8 sectors, one thus obtains

$$\begin{aligned}
&(\mathrm{ee}) : u = (\pm u_1, \cdots, \pm u_p)\,; && u = \left(\pm u_1, \cdots, \pm u_{p-2}, 0, \frac{1}{2}, \frac{1+\tau}{2}, \frac{\tau}{2}\right) \\
&(\mathrm{eo}) : u = \left(\pm u_1, \cdots, \pm u_{p-1}, 0, \frac{\tau}{2}\right); && u = \left(\pm u_1, \cdots, \pm u_{p-1}, \frac{1}{2}, \frac{1+\tau}{2}\right) \\
&(\mathrm{oe}) : u = \left(\pm u_1, \cdots, \pm u_{p-1}, 0, \frac{1}{2}\right); && u = \left(\pm u_1, \cdots, \pm u_{p-1}, 0, \frac{\tau}{2}, \frac{1+\tau}{2}, \frac{\tau}{2}\right) \\
&(\mathrm{oo}) : u = \left(\pm u_1, \cdots, \pm u_{p-1}, 0, \frac{1+\tau}{2}\right); && u = \left(\pm u_1, \cdots, \pm u_{p-1}, \frac{\tau}{2}, \frac{1}{2}\right)
\end{aligned} \tag{3.6}$$

for $O(2p)$, and

$$\begin{aligned}
&(\mathrm{ee}) : u = (\pm u_1, \cdots, \pm u_p, 0)\,; && u = \left(\pm u_1, \cdots, \pm u_{p-1}, \frac{1}{2}, \frac{1+\tau}{2}, \frac{\tau}{2}\right) \\
&(\mathrm{eo}) : u = \left(\pm u_1, \cdots, \pm u_p, \frac{\tau}{2}\right); && u = \left(\pm u_1, \cdots, \pm u_{p-1}, \frac{1}{2}, \frac{1+\tau}{2}, 0\right) \\
&(\mathrm{oe}) : u = \left(\pm u_1, \cdots, \pm u_p, \frac{1}{2}\right); && u = \left(\pm u_1, \cdots, \pm u_{p-1}, \frac{\tau}{2}, \frac{1+\tau}{2}, 0\right) \\
&(\mathrm{oo}) : u = \left(\pm u_1, \cdots, \pm u_p, \frac{1+\tau}{2}\right); && u = \left(\pm u_1, \cdots, \pm u_{p-1}, 0, \frac{\tau}{2}, \frac{1}{2}\right)
\end{aligned} \tag{3.7}$$

for $O(2p+1)$. These u couple minimally to the matters in the fundamental representation. The parameters coupling to a field in a different representation of SO(n) are given by $\rho(u)$, where ρ runs over the weights of the representation of the field.

With the Wilson line backgrounds identified, we study the subgroup of $O(n)$ gauge symmetry which acts within the U_1, U_2 specified above. This is the 'Weyl group,' defined in each disconnected sector of (U_1, U_2). When U_1, U_2 are given by r 2×2 blocks and an $s\times s$ diagonal matrix with ± 1 eigenvalues (with $2r + s = n$ and $s \leq 4$), the Weyl group is given by

$$[\text{Weyl group of } O(2r)] \times [O(s) \text{ elements commuting with the } s\times s \text{ block}]\,. \tag{3.8}$$

The former part has order $2^r r!$, and the latter has order 2^s coming from the $O(s)$ transformations $\mathrm{diag}_{s\times s}(\pm 1, \pm 1, \cdots, \pm 1)$. So the order of the Weyl group $W(O(n))_s$, acting within a given connected sector of U_1, U_2, is given by

$$\begin{aligned}
&|W(O(2p))_0| = 2^p p!\,, \quad |W(O(2p))_2| = 2^{p+1}(p-1)!\,, \quad |W(O(2p))_4| = 2^{p+2}(p-2)! \\
&|W(O(2p+1))_1| = 2^{p+1}p!\,, \quad |W(O(2p+1)_3)| = 2^{p+2}(p-1)!\,,
\end{aligned} \tag{3.9}$$

where the subscript denotes the value of s for U_1, U_2.

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In the above background, the Gaussian path integral of non-zero modes yields $Z_{\text{1-loop}}$, which is the product of the following 1-loop determinants for various supermultiplets [24]:[2]

$$\begin{aligned} Z_{\text{sym. hyper}} &= \prod_{\rho\in\text{sym}} \frac{i\eta(\tau)}{\theta_1(\tau,\epsilon_1+\rho(u))}\cdot\frac{i\eta(\tau)}{\theta_1(\tau,\epsilon_2+\rho(u))} \\ Z_{\text{SO(16) Fermi}} &= \prod_{\rho\in\text{fund}}\prod_{l=1}^{8}\frac{\theta_1(\tau,m_l+\rho(u))}{i\eta(\tau)} \\ Z_{\text{vector}} &= \prod_{i=1}^{r}\left(\frac{2\pi\eta^2 du_i}{i}\cdot\frac{\theta_1(\epsilon_1+\epsilon_2)}{i\eta}\right)\cdot\prod_{\alpha\in\text{root}}\frac{\theta_1(\alpha(u))\theta_1(\epsilon_1+\epsilon_2+\alpha(u))}{i^2\eta^2}\,. \end{aligned} \tag{3.10}$$

Whenever we omit the modular parameters, like $\theta_i(\tau,z)\to\theta_i(z)$ or $\eta(\tau)\to\eta$, it is understood as τ. See appendix A for explanations on these functions. The 'rank' r is the number of continuous complex parameters in U_1, U_2. α runs over the roots of SO(n). Multiplying all these factors, one finally has to integrate over the continuous parameters in u and then sum over disconnected sectors of flat connections. The result is

$$\sum_a \frac{1}{|W_a|}\cdot\frac{1}{(2\pi i)^r}\oint Z^{(a)}_{\text{1-loop}}\ ,\qquad Z^{(a)}_{\text{1-loop}}\equiv Z^{(a)}_{\text{vector}}Z^{(a)}_{\text{sym. hyper}}Z^{(a)}_{\text{SO(16) Fermi}}\,, \tag{3.11}$$

a labels the disconnected sectors of the flat connection U_1, U_2. The integral is a suitable 'contour integral' over the continuous parameters u, to be explained shortly. W_a is the Weyl group with given U_1, U_2 explained above.

Before proceeding, let us comment on the periodicity of (3.10) in u. Each u_i (for $i = 1,\cdots,r$) lives on $T^2/\mathbb{Z}_2$, due to large gauge transformations on T^2, so is a periodic variable $u_i\sim u_i+1\sim u_i+\tau$. However, since $\theta_1(u,\tau)$ is only a quasi-periodic function,

$$\theta_1(z+1)=-\theta_1(z),\ \theta_1(z+\tau)=-q^{-1/2}y^{-1}\theta_1(z),\ \theta_1(z+1+\tau)=q^{-1/2}y^{-1}\theta_1(z)\,, \tag{3.12}$$

with $y\equiv e^{2\pi iz}$, each $\frac{\theta_1}{\eta}$ factor in (3.10) is not invariant under these shifts. The failure of periodicity is related to the gauge anomaly of the chiral theory. The factors spoiling the periodicity cancel in the combination (3.11), due to the anomaly cancelation of our gauge theory.

Another subtlety is the determinant of the real scalars and Majorana fermions. Each real scalar or fermion contributes to a 'square-root' of θ_1 factor. Equivalently, each charge conjugate pair of fermion modes contributes a factor of $\frac{\theta_1(z)}{i\eta}$, while such a pair of bosons contributes $\frac{i\eta}{\theta_1(z)}$ in (3.10). In particular, on these modes, the discrete shifts on the holonomy (3.6), (3.7) given by $u_i=\frac{1}{2},\ \frac{1+\tau}{2},\ \frac{\tau}{2}$ has to be understood with some care. When such a shift is made in the argument of θ_1 coming from a pair of real fields, one should understand it as "$\theta_1(z+u_i)$" $\sim\sqrt{\theta_1(z+u_i)\theta_1(z-u_i)}$. Having this in mind, and applying

$$\theta_1(z+\tfrac{1}{2})=\theta_2(z)\,,\ \ \theta_1(z+\tfrac{\tau}{2})=iq^{-1/8}y^{-1/2}\theta_4(z)\,,\ \ \theta_1(z+\tfrac{1+\tau}{2})=q^{-1/8}y^{-1/2}\theta_3(z)\,, \tag{3.13}$$

[2]One difference from [24] is that we put a factor i in the denominator of the contribution $\frac{\theta_1(q,z)}{i\eta(q)}$ from each Fermi multiplet. Of course this only affects the overall sign of the index, which is ambiguous in 2d without knowing the spin-statistics relation inherited from higher dimensional physics. We shall see that our choice is compatible with the physics of circle compactified 6d CFT, by comparing with some known results. Collecting all the factors of i in $Z_{\text{1-loop}}$, one obtains $(-1)^n$.

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one can replace $\theta_1(z+\frac{1}{2})$, $\theta_1(z+\frac{\tau+1}{2})$, $\theta_1(z+\frac{\tau}{2})$ by $\theta_2(z)$, $\theta_3(z)$, $\theta_4(z)$, respectively, apart from the extra factors appearing in (3.13). These extra factors in (3.11) again cancel to 1. So the theta function θ_1 with a half-period shift can be replaced by one of $\theta_2, \theta_3, \theta_4$ without the shift.

Now we finally explain the meaning of the 'contour integral' in (3.11), following [23, 24]. The 'contour integral' is defined by providing a prescription for the residue sum which replaces the integral, whenever one encounters a pole on the parameter space of (U_1, U_2). The prescription is derived in [24], using the so-called Jeffrey-Kirwan residues. At each pole $u = u_*$ on the r complex dimensional u space, there are r or more hyperplanes of the form $\rho_i(u) + z_i = 0 \pmod{\mathbb{Z} + \tau\mathbb{Z}}$ which passes through it, where $i = 1, \cdots, d$ $(\geq r)$. z_i is a linear combination of the chemical potentials that appears in $\theta_1(\rho_i(u) + z_i)$ in the denominator of $Z_{\text{1-loop}}$. In our problem, z_i is either ϵ_1 or ϵ_2. When exactly r hyperplanes intersect at a point $u = u_*$ (mod $\mathbb{Z} + \tau\mathbb{Z}$), this pole is called non-degenerate. When $d > r$, the pole is called degenerate.

Before explaining the Jeffrey-Kirwan residues (or JK-Res) of our integrand at $u = u_*$, let us first note that the results of [24] apply when the pole at u_* is 'projective.' The pole is called projective when all the weight vectors ρ_i associated with the hyperplanes meeting at $u = u_*$ are contained in a half space. Namely, the projective condition requires that there is a vector v in the Cartan $\mathfrak{h}$ so that $\rho_i(v) > 0$. Note that all non-degenerate poles are projective. In our problem, even for degenerate poles, one can generally show that all poles should be projective, thus allowing us to use the results of [24]. To see this, first note that

$$\rho_i(u_*) = -z_i + m_i + n_i\tau\,, \tag{3.14}$$

for suitable integers m_i, n_i. In our problem, since ρ_i is chosen among the weight system of the $O(n)$ symmetric representation, it is either $\pm 2e_I$ or $\pm e_I \pm e_J$ with $I, J = 1, \cdots, \left[\frac{n}{2}\right]$. Thus, we can take all m_i, n_i to be either 0 or 1 to find all possible solutions for u_*, mod $\mathbb{Z} + \tau\mathbb{Z}$. Also, z_i is either ϵ_1 or ϵ_2 for all i's. Then, taking a solution $u_*(\epsilon_1, \epsilon_2)$ which depends on $\epsilon_{1,2}$, one deforms the solution to the regime in which ϵ_1, ϵ_2 are real and negative, taken to be $-\epsilon_{1,2} \gg 1$ and $-\epsilon_{1,2} \gg |\mathrm{Re}(\tau)|$. Then one finds that $\rho_i \cdot \mathrm{Re}(u_*) > 0$, fulfilling the projective condition. In fact, one can always provide this kind of argument on the projectivity of poles when the system has independent flavor symmetry for each matter supermultiplet. The $\mathcal{N} = (2,2)$ or $(0,2)$ models may exhibit non-projective poles if there are nonzero superpotentials so that flavor symmetries are restricted. In $\mathcal{N} = (0,4)$ models, independent flavor symmetry can be found for each hypermultiplet. This is why it is easier to apply the results of [24] to $(0,4)$ theories. For instance, the quantum mechanical version of this index formula is well applicable to the ADHM instanton quantum mechanics [34], as these systems always have $(0,4)$ SUSY. (The results of [34] will be used in our section 4.)

[24] finds that the integral in (3.11) is given by

$$\frac{1}{(2\pi i)^r}\oint Z^{(a)}_{\text{1-loop}} = \sum_{u_*} \text{JK-Res}_{u_*}(\mathbf{Q}_*, \eta) Z^{(a)}_{\text{1-loop}}\,, \tag{3.15}$$

where u_* runs over all the poles in the integrand. The JK-Res appearing in this expression is given as follows. JK-Res is a linear functional which refers to an auxiliary vector η in the charge space, and also to the set of charge vectors $\mathbf{Q}_* = (Q_1, \cdots, Q_d)$ for the hyperplanes crossing u_*. The defining property of $\text{JK-Res}_{u_*}(\mathbf{Q}_*, \eta)$ is

$$\text{JK-Res}_{u_*}(\mathbf{Q}_*,\eta)\frac{dQ_{j_1}(u)\wedge\cdots\wedge dQ_{j_r}(u)}{Q_{j_1}(u-u_*)\cdots Q_{j_r}(u-u_*)} = \begin{cases} \text{sign}\det(Q_{j_1},\cdots,Q_{j_r}) & \text{if } \eta\in\text{Cone}(Q_{j_1},\cdots,Q_{j_r}) \\ 0 & \text{otherwise} \end{cases}, \tag{3.16}$$

or equivalently

$$\text{JK-Res}_{u_*}(\mathbf{Q}_*,\eta)\frac{du_1\wedge\cdots\wedge du_r}{Q_{j_1}(u-u_*)\cdots Q_{j_r}(u-u_*)} = \begin{cases} |\det(Q_{j_1},\cdots,Q_{j_r})|^{-1} & \text{if } \eta\in\text{Cone}(Q_{j_1},\cdots,Q_{j_r}) \\ 0 & \text{otherwise} \end{cases}. \tag{3.17}$$

To make the condition $\eta \in \text{Cone}(Q_{j_1}, \cdots, Q_{j_r})$ unambiguous, one has to put η at a sufficiently generic point, as explained in [24]. These rules are giving a definite residue when the integrand takes the form of a 'simple pole.' Although this definition apparently overdetermines JK-Res due to many relations among the forms $\bigwedge_{i=1}^{r}\frac{dQ_{j_i}(u)}{Q_{j_i}(u)}$, it turns out to be consistent (see [24] and references therein). As one expands the integrand $Z^{(a)}_{\text{1-loop}}$ around $u = u_*$, one will encounter not just simple poles, but also multiple poles and less singular homogeneous expressions in $u - u_*$, multiplied by $du_1 \wedge \cdots \wedge du_r$. The JK-Res of the last two classes of monomials are all (naturally) zero: this is also consistent with the alternative 'constructive definition,' which expresses JK-Res as an iterated integral over a cycle. Using this definition to compute the integral is especially simple for non-degenerate poles, in which case one can directly read off a unique integral of the form (3.17) at a given $u = u_*$. The case with degenerate poles require some more work, but of course coming with a clear rule. The final result (3.15) is independent of the choice of η [24].

In the remaining part of this section, we first analyze the elliptic genera for $n = 1, 2, 3, 4$ E-strings in great detail. In section 3.5, we then illustrate the structure of the higher E-string indices. In particular, degenerate poles start to appear from $n \geq 6$. The residue evaluations are almost as simple as the non-degenerate poles for $n = 6, 7$, all coming from simple poles. Their residues are simply given by combinations of theta functions. For $n \geq 8$, we explain that there start to appear degenerate poles which are also multiple poles. Their residues are given by theta functions and their derivatives in the elliptic parameters.

3.1 One E-string

We consider the elliptic genus for the $O(1)$ theory. Since $O(1) = \mathbb{Z}_2$, there are four different flat connections $(1, 1)$, $(1, -1)$, $(-1, 1)$, $(-1, -1)$. The indices in the four sectors are given by

$$Z_{1(i)} = -[1]_{\text{vec}} \cdot \left[\frac{\eta^2}{\theta_1(\epsilon_1)\theta_1(\epsilon_2)}\right]_{\text{sym hyper}} \cdot \left[\prod_{l=1}^{8}\frac{\theta_i(m_l)}{\eta}\right]_{\text{Fermi}}, \tag{3.18}$$

where $i = 1, 2, 3, 4$ for the Wilson line $(1, 1), (-1, 1), (-1, -1), (1, -1)$, respectively. Combining all four contributions, and dividing by the Weyl group order $|W| = 2$ in each sector, the full index is given by

$$Z_1 = \sum_{i=1}^{4}\frac{Z_{1(i)}}{2} = -\frac{\Theta(q, m_l)}{\eta^6\theta_1(\epsilon_1)\theta_1(\epsilon_2)}, \tag{3.19}$$

where the E_8 theta function Θ is given by

$$\Theta(\tau, m_l) = \frac{1}{2}\sum_{n=1}^{4}\prod_{l=1}^{8}\theta_n(\tau, m_l)\,. \tag{3.20}$$

Physically, $\frac{Z_{1(1)}+Z_{1(2)}}{2}$ simply imposes the $O(1) = \mathbb{Z}_2$ singlet condition, while the remainder $\frac{Z_{1(3)}+Z_{1(4)}}{2}$ is the contribution from the twisted sector.

In [19], the above result was derived using topological strings and was explained using an effective free string theory calculus, in which the left moving sector consists of the E_8 current algebra at level 1 and the right moving sector consists of a $(0, 4)$ supersymmetric string with target space $\mathbb{R}^4$. The four terms of $\Theta(\tau, m_i)$ can be understood as coming from the Ramond and Neveu-Schwarz sectors of the left-moving fermions, and then truncating the Hilbert space by a GSO projection. In our UV gauge theory calculus, the twisting and GSO projection come from the $O(1)$ gauge symmetry. These summation and projection will generalize curiously to higher $O(n)$ gauge theories below. It will be interesting to see if one can provide a CFT interpretation, extending the notions of twisted sectors and GSO projection.

Since $\Theta(q, m_l)$ is given by the summation over the E_8 root lattice, Z_1 has a manifest E_8 symmetry, and is expanded as the sum of E_8 characters. This supports the IR enhancement SO(16) $\to E_8$ of global symmetry in our gauge theory.

3.2 Two E-strings

Now we consider the $O(2)$ theory. There are 7 sectors of $O(2)$ Wilson lines given by (3.5). One in the (ee) sector has a complex modulus, while the other six are all discrete. We name the sectors as follows, where (a_+, a_-) are the two eigenvalues of u in the discrete sectors which act on the fundamental representation [23]:

$$\begin{aligned}
(0) \equiv (\text{ee}) \quad &: \qquad\qquad (U_1, U_2) = (e^{iu_1\sigma_2}, e^{iu_2\sigma_2}) \\
(1), (2) \equiv (\text{oe})_\pm &: \ (\sigma_3, \pm 1) \to (a_v, a_+, a_-) = (\tfrac{1}{2}, 0, \tfrac{1}{2})\,, \quad (\tfrac{1}{2}, \tfrac{\tau}{2}, \tfrac{1+\tau}{2}) \\
(3), (4) \equiv (\text{eo})_\pm &: \ (\pm 1, \sigma_3) \to (a_v, a_+, a_-) = (\tfrac{\tau}{2}, 0, \tfrac{\tau}{2})\,, \quad (\tfrac{\tau}{2}, \tfrac{1}{2}, \tfrac{1+\tau}{2}) \\
(5), (6) \equiv (\text{oo})_\pm &: (\pm\sigma_3, \sigma_3) \to (a_v, a_+, a_-) = (\tfrac{1+\tau}{2}, 0, \tfrac{1+\tau}{2})\,, \ (\tfrac{1+\tau}{2}, \tfrac{1}{2}, \tfrac{\tau}{2})\,.
\end{aligned}$$

All eigenvalues a_+, a_- are defined mod $\mathbb{Z}+\tau\mathbb{Z}$. $a_v = a_+ + a_-$ is the eigenvalue acting on the $O(2)$ adjoint (antisymmetric) representation. The discrete holonomy eigenvalues acting on the $O(2)$ symmetric representation are $a_v = a_+ + a_-$, $2a_+$, $2a_-$. The contributions $Z_{2(a)}$ (with $a = 0, \cdots, 6$) are given by

$$\begin{aligned}
Z_{2(0)} &= \oint \left[\eta^2 du \cdot \frac{\theta_1(2\epsilon_+)}{i\eta}\right]_{\text{vec}} \cdot \left[\frac{\eta^6}{\theta_1(\epsilon_1)\theta_1(\epsilon_2)\theta_1(\epsilon_1 \pm 2u)\theta_1(\epsilon_2 \pm 2u)}\right]_{\text{sym}} \cdot \left[\prod_{l=1}^{8}\frac{\theta_1(m_l \pm u)}{\eta^2}\right]_{\text{Fermi}} \\
Z_{2(a)} &= \left[\frac{\theta_1(a_v)\theta_1(2\epsilon_+ + a_v)}{\eta^2}\right]_{\text{vec}} \cdot \left[\frac{\eta^6}{\theta_1(\epsilon_1 + a_v)\theta_1(\epsilon_2 + a_v)\theta_1(\epsilon_1 + 2a_\pm)\theta_1(\epsilon_2 + 2a_\pm)}\right]_{\text{sym}} \\
&\quad \cdot \left[\prod_{l=1}^{8}\frac{\theta_1(m_l + a_+)\theta_1(m_l + a_-)}{\eta^2}\right]_{\text{Fermi}} \qquad (a = 1, \cdots, 6)\,,
\end{aligned} \tag{3.21}$$

where we defined $\epsilon_+ = \frac{\epsilon_1+\epsilon_2}{2}$. As explained after (3.13), $\theta_1(z + a_v)$ factors should be understood as θ_i, with $i = 1, 2, 3, 4$ for $a_v = 0, \frac{1}{2}, \frac{1+\tau}{2}, \frac{\tau}{2}$, respectively.

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The contour integral in $Z_{2(0)}$ can be done by taking residues from poles with positive SO(2) electric charge only: this is the simple rule for the rank 1 theory obtained by taking $\eta = 1$ [23]. The relevant poles are at $\theta_1(\epsilon_1 + 2u) = 0$ and $\theta_1(\epsilon_2 + 2u) = 0$. Using

$$\frac{1}{2\pi i}\oint_{u=a+b\tau}\frac{du}{\theta_1(\tau|u)} = \frac{(-1)^{a+b}e^{i\pi b^2\tau}}{\theta_1'(\tau|0)} = \frac{(-1)^{a+b}e^{i\pi b^2\tau}}{2\pi\eta^3}\,, \tag{3.22}$$

one should pick the residues at $u = -\frac{\epsilon_{1,2}}{2}, -\frac{\epsilon_{1,2}}{2} + \frac{1}{2}, -\frac{\epsilon_{1,2}}{2} + \frac{1+\tau}{2}, -\frac{\epsilon_{1,2}}{2} + \frac{\tau}{2}$. The residue sum is

$$Z_{2(0)} = \frac{1}{2\eta^{12}\theta_1(\epsilon_1)\theta_1(\epsilon_2)}\sum_{i=1}^{4}\left[\frac{\prod_{l=1}^{8}\theta_i(m_l \pm \frac{\epsilon_1}{2})}{\theta_1(2\epsilon_1)\theta_1(\epsilon_2-\epsilon_1)} + \frac{\prod_{l=1}^{8}\theta_i(m_l \pm \frac{\epsilon_2}{2})}{\theta_1(2\epsilon_2)\theta_1(\epsilon_1-\epsilon_2)}\right]. \tag{3.23}$$

Expressions with $\pm$ signs mean $\theta_i(x \pm y) \equiv \theta_i(x+y)\theta_i(x-y)$. The contributions from the other six sectors are

$$\begin{aligned}
Z_{2(1)} &= \frac{\theta_2(0)\theta_2(2\epsilon_+)\prod_{l=1}^{8}\theta_1(m_l)\theta_2(m_l)}{\eta^{12}\theta_1(\epsilon_1)^2\theta_1(\epsilon_2)^2\theta_2(\epsilon_1)\theta_2(\epsilon_2)}\,, \quad & Z_{2(2)} &= \frac{\theta_2(0)\theta_2(2\epsilon_+)\prod_{l=1}^{8}\theta_3(m_l)\theta_4(m_l)}{\eta^{12}\theta_1(\epsilon_1)^2\theta_1(\epsilon_2)^2\theta_2(\epsilon_1)\theta_2(\epsilon_2)}\,,\\
Z_{2(3)} &= \frac{\theta_4(0)\theta_4(2\epsilon_+)\prod_{l=1}^{8}\theta_1(m_l)\theta_4(m_l)}{\eta^{12}\theta_1(\epsilon_1)^2\theta_1(\epsilon_2)^2\theta_4(\epsilon_1)\theta_4(\epsilon_2)}\,, \quad & Z_{2(4)} &= \frac{\theta_4(0)\theta_4(2\epsilon_+)\prod_{l=1}^{8}\theta_2(m_l)\theta_3(m_l)}{\eta^{12}\theta_1(\epsilon_1)^2\theta_1(\epsilon_2)^2\theta_4(\epsilon_1)\theta_4(\epsilon_2)}\,,\\
Z_{2(5)} &= \frac{\theta_3(0)\theta_3(2\epsilon_+)\prod_{l=1}^{8}\theta_1(m_l)\theta_3(m_l)}{\eta^{12}\theta_1(\epsilon_1)^2\theta_1(\epsilon_2)^2\theta_3(\epsilon_1)\theta_3(\epsilon_2)}\,, \quad & Z_{2(6)} &= \frac{\theta_3(0)\theta_3(2\epsilon_+)\prod_{l=1}^{8}\theta_2(m_l)\theta_4(m_l)}{\eta^{12}\theta_1(\epsilon_1)^2\theta_1(\epsilon_2)^2\theta_3(\epsilon_1)\theta_3(\epsilon_2)}\,.
\end{aligned} \tag{3.24}$$

The two E-string elliptic genus is given by

$$Z_2(\tau,\epsilon_{1,2},m_l) = \frac{1}{2}Z_{2(0)} + \frac{1}{4}\sum_{a=1}^{6}Z_{2(a)}\,, \tag{3.25}$$

dividing each $Z_{2(a)}$ by the order of the 'Weyl group,' given by (3.9).

Recently, [7] obtained the 2 E-string elliptic genus. This was done by constraining its form with its modularity, the 'domain wall' ansatz of [5], and a few low order coefficients in the genus expansion known from the topological string calculus. The result of [7] is given by

$$\begin{aligned}
Z_2 = {} & \frac{1}{576\eta^{12}\theta_1(\epsilon_1)\theta_1(\epsilon_2)\theta_1(\epsilon_2-\epsilon_1)\theta_1(2\epsilon_1)}\left[4A_1^2(\phi_{0,1}(\epsilon_1)^2 - E_4\theta_{-2,1}(\epsilon_1)^2)\right. \\
& \left. + 3A_2(E_4^2\phi_{-2,1}(\epsilon_1)^2 - E_6\phi_{-2,1}(\epsilon_1)\phi_{0,1}(\epsilon_1)) + 5B_2(E_6\phi_{-2,1}(\epsilon_1)^2 - E_4\phi_{-2,1}(\epsilon_1)\phi_{0,1}(\epsilon_1))\right] \\
& + (\epsilon_1 \leftrightarrow \epsilon_2)
\end{aligned} \tag{3.26}$$

where $E_4(\tau)$, $E_6(\tau)$ are the Eisenstein series, summarized in appendix A,

$$\phi_{-2,1}(\epsilon,\tau) = -\frac{\theta_1(\epsilon,\tau)^2}{\eta(\tau)^6}\,, \quad \phi_{0,1}(\epsilon,\tau) = 4\left[\frac{\theta_2(\epsilon,\tau)^2}{\theta_2(0,\tau)^2} + \frac{\theta_3(\epsilon,\tau)^2}{\theta_3(0,\tau)^2} + \frac{\theta_4(\epsilon,\tau)^2}{\theta_4(0,\tau)^2}\right], \tag{3.27}$$

and $A_1(m_l)$, $A_2(m_l)$, $B_2(m_l)$ are three of the nine Jacobi forms which are invariant under the Weyl group of E_8. See, for instance, the appendix of [29] for the full list. A_1 is simply the E_8 theta function $A_1 = \Theta(m_l,\tau)$, and

$$\begin{aligned}
A_2 &= \frac{8}{9}\left[\Theta(2m_l,2\tau) + \frac{\Theta(m_l,\frac{\tau}{2}) + \Theta(m_l,\frac{\tau+1}{2})}{16}\right] \\
B_2 &= \frac{8}{15}\left[(\theta_3^4+\theta_4^4)\Theta(2m_l,2\tau) - \frac{1}{16}(\theta_2^4+\theta_3^4)\Theta(m_l,\tfrac{\tau}{2}) + \frac{1}{16}(\theta_2^4-\theta_4^4)\Theta(m_l,\tfrac{\tau+1}{2})\right],
\end{aligned} \tag{3.28}$$

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where $\theta_i \equiv \theta_i(0)$. We made a full analytic proof, at $\epsilon_1 = -\epsilon_2$ for simplicity (but keeping all E_8 masses and $\epsilon_- = \frac{\epsilon_1-\epsilon_2}{2}$), that (3.25) and (3.26) agree with each other. See appendix C for our proof. On one side, this agreement shows that the 'domain wall ansatz' of [7] is at work. On the other hand, it also shows that our gauge theory index exhibits the Weyl symmetry of E_8, which is manifest in (3.26). So this supports the IR E_8 symmetry enhancement of our gauge theory.

3.3 Three E-strings

There are eight sectors of $O(3)$ holonomies on T^2, which we label as follows:

$$\begin{aligned}
&(\mathrm{ee}): \mathrm{diag}(e^{iu_1\sigma_2},1), \ \ \mathrm{diag}(e^{iu_2\sigma_2},1) \ \ \to (1)\,; \ \ \mathrm{diag}(-1,-1,1), \mathrm{diag}(1,-1,-1) \to (1)'\,;\\
&(\mathrm{eo}): \mathrm{diag}(e^{iu_1\sigma_2},1), \ \ \mathrm{diag}(e^{iu_2\sigma_2},-1) \to (4)\,; \ \ \mathrm{diag}(-1,-1,1), \mathrm{diag}(1,-1,1) \ \ \to (4)'\,;\\
&(\mathrm{oe}): \mathrm{diag}(e^{iu_1\sigma_2},-1), \mathrm{diag}(e^{iu_2\sigma_2},1) \ \ \to (2)\,; \ \ \mathrm{diag}(1,-1,1), \ \ \mathrm{diag}(-1,-1,1) \to (2)'\,;\\
&(\mathrm{oo}): \mathrm{diag}(e^{iu_1\sigma_2},-1), \mathrm{diag}(e^{iu_2\sigma_2},-1) \to (3)\,; \ \ \mathrm{diag}(1,1,-1), \ \ \mathrm{diag}(1,-1,1) \ \ \to (3)'\,.
\end{aligned}$$

The indices in various sectors are given as follows. Firstly,

$$\begin{aligned}
Z_{3(1)} =& -\oint \left[\eta^2 du \cdot \frac{\theta_1(2\epsilon_+)\theta_1(2\epsilon_+ \pm u)\theta_1(\pm u)}{i\eta^5}\right]_{\rm vec} \cdot \left[\frac{\eta^{12}}{\theta_1(\epsilon_{1,2})^2\theta_1(\epsilon_{1,2}\pm u)\theta_1(\epsilon_{1,2}\pm 2u)}\right]_{\rm sym}\\
&\cdot \left[\prod_{l=1}^{8} \frac{\theta_1(m_l)\theta_1(m_l+u)\theta_1(m_l-u)}{\eta^3}\right]_{\rm Fermi} \qquad (3.29)\\
Z_{3(1)'} =& -\left[\frac{\theta_2(0)\theta_3(0)\theta_4(0)\theta_2(2\epsilon_+)\theta_3(2\epsilon_+)\theta_4(2\epsilon_+)}{\eta^6}\right]_{\rm vec} \cdot \left[\frac{\eta^{12}}{\theta_1(\epsilon_{1,2})^3\theta_2(\epsilon_{1,2})\theta_3(\epsilon_{1,2})\theta_4(\epsilon_{1,2})}\right]_{\rm sym}\\
&\cdot \left[\prod_{l=1}^{8} \frac{\theta_2(m_l)\theta_3(m_l)\theta_4(m_l)}{\eta^3}\right]_{\rm Fermi} . \qquad (3.30)
\end{aligned}$$

$Z_{3(1)'}$ is obtained with discrete holonomy $(a_1, a_2, a_3) = (\frac{1}{2}, \frac{1+\tau}{2}, \frac{\tau}{2})$ acting on the fundamental, $(a_1+a_2, a_2+a_3, a_3+a_1) = (\frac{\tau}{2}, \frac{1}{2}, \frac{1+\tau}{2})$ on adjoint, and $(2a_1, 2a_2, 2a_3, a_1+a_2, a_2+a_3, a_3+a_1)$ on symmetric representations. Similarly, one obtains

$$\begin{aligned}
Z_{3(4)} =& -\oint \left[\eta^2 du \cdot \frac{\theta_1(2\epsilon_+)\theta_4(2\epsilon_+ \pm u)\theta_4(\pm u)}{i\eta^5}\right]_{\rm vec} \cdot \left[\frac{\eta^{12}}{\theta_1(\epsilon_{1,2})^2\theta_4(\epsilon_{1,2}\pm u)\theta_1(\epsilon_{1,2}\pm 2u)}\right]_{\rm sym}\\
&\cdot \left[\prod_{l=1}^{8} \frac{\theta_4(m_l)\theta_1(m_l+u)\theta_1(m_l-u)}{\eta^3}\right]_{\rm Fermi} \qquad (3.31)\\
Z_{3(4)'} =& -\left[\frac{\theta_2(0)\theta_3(0)\theta_4(0)\theta_2(2\epsilon_+)\theta_3(2\epsilon_+)\theta_4(2\epsilon_+)}{\eta^6}\right]_{\rm vec} \cdot \left[\frac{\eta^{12}}{\theta_1(\epsilon_{1,2})^3\theta_2(\epsilon_{1,2})\theta_3(\epsilon_{1,2})\theta_4(\epsilon_{1,2})}\right]_{\rm sym}\\
&\cdot \left[\prod_{l=1}^{8} \frac{\theta_1(m_l)\theta_2(m_l)\theta_3(m_l)}{\eta^3}\right]_{\rm Fermi} \qquad (3.32)
\end{aligned}$$

from the (eo) sectors with $(a_1,a_2,a_3)=(\frac{1}{2},\frac{1+\tau}{2},0)$ for $Z_{3(4)'}$,

$$Z_{3(2)} = -\oint\left[\eta^2 du\cdot\frac{\theta_1(2\epsilon_+)\theta_2(2\epsilon_+\pm u)\theta_2(\pm u)}{i\eta^5}\right]_{\text{vec}}\cdot\left[\frac{\eta^{12}}{\theta_1(\epsilon_{1,2})^2\theta_2(\epsilon_{1,2}\pm u)\theta_1(\epsilon_{1,2}\pm 2u)}\right]_{\text{sym}}$$
$$\cdot\left[\prod_{l=1}^{8}\frac{\theta_2(m_l)\theta_1(m_l+u)\theta_1(m_l-u)}{\eta^3}\right]_{\text{Fermi}} \tag{3.33}$$

$$Z_{3(2)'} = -\left[\frac{\theta_2(0)\theta_3(0)\theta_4(0)\theta_2(2\epsilon_+)\theta_3(2\epsilon_+)\theta_4(2\epsilon_+)}{\eta^6}\right]_{\text{vec}}\cdot\left[\frac{\eta^{12}}{\theta_1(\epsilon_{1,2})^3\theta_2(\epsilon_{1,2})\theta_3(\epsilon_{1,2})\theta_4(\epsilon_{1,2})}\right]_{\text{sym}}$$
$$\cdot\left[\prod_{l=1}^{8}\frac{\theta_1(m_l)\theta_3(m_l)\theta_4(m_l)}{\eta^3}\right]_{\text{Fermi}} \tag{3.34}$$

from the (oe) sectors with $(a_1,a_2,a_3)=(\frac{\tau}{2},\frac{1+\tau}{2},0)$ for $Z_{3(2)'}$, and

$$Z_{3(3)} = -\oint\left[\eta^2 du\cdot\frac{\theta_1(2\epsilon_+)\theta_3(2\epsilon_+\pm u)\theta_3(\pm u)}{i\eta^5}\right]_{\text{vec}}\cdot\left[\frac{\eta^{12}}{\theta_1(\epsilon_{1,2})^2\theta_3(\epsilon_{1,2}\pm u)\theta_1(\epsilon_{1,2}\pm 2u)}\right]_{\text{sym}}$$
$$\cdot\left[\prod_{l=1}^{8}\frac{\theta_3(m_l)\theta_1(m_l+u)\theta_1(m_l-u)}{\eta^3}\right]_{\text{Fermi}} \tag{3.35}$$

$$Z_{3(3)'} = -\left[\frac{\theta_2(0)\theta_3(0)\theta_4(0)\theta_2(2\epsilon_+)\theta_3(2\epsilon_+)\theta_4(2\epsilon_+)}{\eta^6}\right]_{\text{vec}}\cdot\left[\frac{\eta^{12}}{\theta_1(\epsilon_{1,2})^3\theta_2(\epsilon_{1,2})\theta_3(\epsilon_{1,2})\theta_4(\epsilon_{1,2})}\right]_{\text{sym}}$$
$$\cdot\left[\prod_{l=1}^{8}\frac{\theta_1(m_l)\theta_2(m_l)\theta_4(m_l)}{\eta^3}\right]_{\text{Fermi}} \tag{3.36}$$

from the (oo) sectors with $(a_1,a_2,a_3)=(0,\frac{\tau}{2},\frac{1}{2})$ for $Z_{3(3)'}$. The contour integrals in $Z_{3(i)}$ acquire residue contributions from poles $u_*=-\frac{\epsilon_{1,2}}{2},-\frac{\epsilon_{1,2}}{2}+\frac{1}{2},-\frac{\epsilon_{1,2}}{2}+\frac{\tau}{2},-\frac{\epsilon_{1,2}}{2}+\frac{1+\tau}{2}$ and $u_*=-\epsilon_{1,2}+\cdots$, where $\cdots$ part is decided by $\theta_i(u+\epsilon_{1,2})=0$. The residue sums are given by

$$Z_{3(i)} = -\frac{\eta^4}{\theta_1(\epsilon_1)^2\theta_1(\epsilon_2)^2}\left[\frac{\eta^2\theta_1(\epsilon_1)\theta_1(\epsilon_2)}{\theta_1(2\epsilon_1)\theta_1(\epsilon_2-\epsilon_1)\theta_1(3\epsilon_1)\theta_1(\epsilon_2-2\epsilon_1)}\prod_{l=1}^{8}\frac{\theta_i(m_l)\theta_i(m_l\pm\epsilon_1)}{\eta^3}\right. \tag{3.37}$$
$$\left.+\frac{1}{2}\sum_{a=1}^{4}\frac{\eta^2\theta_{\sigma_i(a)}(\frac{3\epsilon_1}{2}+\epsilon_2)\theta_{\sigma_i(a)}(-\frac{\epsilon_1}{2})}{\theta_1(2\epsilon_1)\theta_1(\epsilon_2-\epsilon_1)\theta_{\sigma_i(a)}(\frac{3\epsilon_1}{2})\theta_{\sigma_i(a)}(\epsilon_2-\frac{\epsilon_1}{2})}\prod_{l=1}^{8}\frac{\theta_i(m_l)\theta_a(m_l\pm\frac{\epsilon_1}{2})}{\eta^3}+(\epsilon_1\leftrightarrow\epsilon_2)\right]$$

where the permutations are defined by

$$\sigma_1(1,2,3,4)=(1,2,3,4)\,,\qquad \sigma_2(1,2,3,4)=(2,1,4,3),$$
$$\sigma_3(1,2,3,4)=(3,4,1,2)\,,\qquad \sigma_4(1,2,3,4)=(4,3,2,1)\,. \tag{3.38}$$

The full index is given by

$$Z_3=\sum_{i=1}^{4}\left(\frac{1}{4}Z_{3(i)}+\frac{1}{8}Z_{3(i)'}\right), \tag{3.39}$$

after dividing by the Weyl factors (3.9).

For simplicity, we study the indices at $m_l = 0$, $\epsilon_1 = -\epsilon_2 \equiv \epsilon$ in more detail, which are

$$Z_{3(i)} = \frac{\eta^4}{\theta_1(\epsilon)^4}\left[\frac{2\theta_1(\epsilon)^2\theta_i(0)^8\theta_i(\epsilon)^{16}}{\eta^{22}\theta_1(2\epsilon)^2\theta_1(3\epsilon)^2} + \sum_{a=1}^{4}\frac{\theta_{\sigma_i(a)}(\frac{\epsilon}{2})^2\theta_i(0)^8\theta_a(\frac{\epsilon}{2})^{16}}{\eta^{22}\theta_1(2\epsilon)^2\theta_{\sigma_i(a)}(\frac{3\epsilon}{2})^2}\right] \tag{3.40}$$

and

$$Z_{3(1)'} = \frac{\theta_2(0)^{10}\theta_3(0)^{10}\theta_4(0)^{10}}{\eta^{18}\theta_1(\epsilon)^6\theta_2(\epsilon)^2\theta_3(\epsilon)^2\theta_4(\epsilon)^2} = \frac{4\theta_2(0)^8\theta_3(0)^8\theta_4(0)^8}{\eta^{18}\theta_1(\epsilon)^4\theta_1(2\epsilon)^2}\,, \tag{3.41}$$

with $Z_{3(2)'} = Z_{3(3)'} = Z_{3(4)'} = 0$. We consider the genus expansion of Z_3, where genus is defined for the topological string amplitudes on the CY_3 which engineers our 6d CFT in the F-theory context. Namely, we expand

$$F_3 \equiv Z_3 - Z_1Z_2 + \frac{1}{3}{Z_1}^3 = \sum_{n\geq 0, g\geq 0}(\epsilon_1+\epsilon_2)^n(\epsilon_1\epsilon_2)^{g-1}F^{(n,g,3)}(\tau)\,. \tag{3.42}$$

Taking $\epsilon_+ = 0$, some known results on $F^{(0,g,3)}$ are summarized in (B.1), which were computed in [43] up to genus 5. This can be compared with $F^{(0,g,3)}$ obtained from our gauge theory index. Numerically, we checked the agreements for $g \leq 5$ up to first 10 terms in the q expansions, starting at $q^{-3/2}$, with the last term that we checked at $q^{15/2}$. (The two coefficients at $q^{-1/2}$ and $q^{1/2}$ are zero, due to a vanishing theorem.)

We also analytically checked the agreements for $F^{(0,0,3)}$, $F^{(0,1,3)}$, and a refined amplitude $F^{(1,0,3)}$, against the results known from the topological string calculus. See appendix C for the details.

3.4 Four E-strings

The indices from the two sectors in the (ee) part of $O(4)$ holonomy are

$$\begin{aligned} Z_{4(1)} = &-\oint\left[\eta^4 du_1du_2\cdot\frac{\theta_1(2\epsilon_+)^2\theta_1(2\epsilon_+\pm u_1\pm u_2)\theta_1(\pm u_1\pm u_2)}{\eta^{10}}\right]_{\text{vec}} \\ &\cdot\left[\frac{\eta^{20}}{\theta_1(\epsilon_{1,2})^2\theta_1(\epsilon_{1,2}\pm u_1\pm u_2)\theta_1(\epsilon_{1,2}\pm 2u_1)\theta_1(\epsilon_{1,2}\pm 2u_2)}\right]_{\text{sym}}\cdot\left[\prod_{l=1}^{8}\frac{\theta_1(m_l\pm u_1)\theta_1(m_l\pm u_2)}{\eta^4}\right]_{\text{Fermi}} \end{aligned} \tag{3.43}$$

$$\begin{aligned} Z_{4(1)'} = &\left[\frac{\theta_2(0)^2\theta_3(0)^2\theta_4(0)^2\theta_2(2\epsilon_+)^2\theta_3(2\epsilon_+)^2\theta_4(2\epsilon_+)^2}{\eta^{12}}\right]_{\text{vec}} \\ &\cdot\left[\frac{\eta^{20}}{\theta_1(\epsilon_{1,2})^4\theta_2(\epsilon_{1,2})^2\theta_3(\epsilon_{1,2})^2\theta_4(\epsilon_{1,2})^2}\right]_{\text{sym}}\cdot\left[\prod_{l=1}^{8}\frac{\theta_1(m_l)\theta_2(m_l)\theta_3(m_l)\theta_4(m_l)}{\eta^4}\right]_{\text{Fermi}} \end{aligned} \tag{3.44}$$

where $Z_{4(1)'}$ is obtained with discrete holonomy $(a_1, a_2, a_3, a_4) = (0, \frac{1}{2}, \frac{1+\tau}{2}, \frac{\tau}{2})$ for the fundamental representation. We used a shorthand notation $\theta_i(\epsilon_{1,2}) \equiv \theta_i(\epsilon_1)\theta_i(\epsilon_2)$. The indices from the two sectors in the (oe) part are

$$\begin{aligned} Z_{4(2)} = &\oint\left[\eta^2 du\cdot\frac{\theta_1(2\epsilon_+)\theta_2(2\epsilon_+)\theta_1(2\epsilon_+\pm u)\theta_2(2\epsilon_+\pm u)\theta_2(0)\theta_1(\pm u)\theta_2(\pm u)}{i\eta^{11}}\right]_{\text{vec}} \\ &\cdot\left[\frac{\eta^{20}}{\theta_1(\epsilon_{1,2}\pm 2u)\theta_1(\epsilon_{1,2})^3\theta_2(\epsilon_{1,2})\theta_1(\epsilon_{1,2}\pm u)\theta_2(\epsilon_{1,2}\pm u)}\right]_{\text{sym}}\cdot\left[\prod_{l=1}^{8}\frac{\theta_1(m_l\pm u)\theta_1(m_l)\theta_2(m_l)}{\eta^4}\right]_{\text{Fermi}} \end{aligned} \tag{3.45}$$

$$\begin{aligned} Z_{4(2)'} = &\oint\left[\eta^2 du\cdot\frac{\theta_1(2\epsilon_+)\theta_2(2\epsilon_+)\theta_3(2\epsilon_+\pm u)\theta_4(2\epsilon_+\pm u)\theta_2(0)\theta_3(\pm u)\theta_4(\pm u)}{i\eta^{11}}\right]_{\text{vec}} \\ &\cdot\left[\frac{\eta^{20}}{\theta_1(\epsilon_{1,2}\pm 2u)\theta_1(\epsilon_{1,2})^3\theta_2(\epsilon_{1,2})\theta_3(\epsilon_{1,2}\pm u)\theta_4(\epsilon_{1,2}\pm u)}\right]_{\text{sym}}\cdot\left[\prod_{l=1}^{8}\frac{\theta_1(m_l\pm u)\theta_3(m_l)\theta_4(m_l)}{\eta^4}\right]_{\text{Fermi}} \end{aligned} \tag{3.46}$$

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where the holonomy $(a_1, a_2, a_3, a_4) = (u, -u, 0, \frac{1}{2})$ and $(u, -u, \frac{\tau}{2}, \frac{1+\tau}{2})$ are used for $Z_{4(2)}$ and $Z_{4(2)'}$, respectively. The indices from the two sectors in the (oo) part are

$$Z_{4(3)} = \oint \left[\eta^2 du \cdot \frac{\theta_1(2\epsilon_+)\theta_3(2\epsilon_+)\theta_1(2\epsilon_+ \pm u)\theta_3(2\epsilon_+ \pm u)\theta_3(0)\theta_1(\pm u)\theta_3(\pm u)}{i\eta^{11}}\right]_{\text{vec}} \cdot \left[\frac{\eta^{20}}{\theta_1(\epsilon_{1,2}\pm 2u)\theta_1(\epsilon_{1,2})^3\theta_3(\epsilon_{1,2})\theta_1(\epsilon_{1,2}\pm u)\theta_3(\epsilon_{1,2}\pm u)}\right]_{\text{sym}} \cdot \left[\prod_{l=1}^{8}\frac{\theta_1(m_l\pm u)\theta_1(m_l)\theta_3(m_l)}{\eta^4}\right]_{\text{Fermi}} \tag{3.47}$$

$$Z_{4(3)'} = \oint \left[\eta^2 du \cdot \frac{\theta_1(2\epsilon_+)\theta_3(2\epsilon_+)\theta_2(2\epsilon_+ \pm u)\theta_4(2\epsilon_+ \pm u)\theta_3(0)\theta_2(\pm u)\theta_4(\pm u)}{i\eta^{11}}\right]_{\text{vec}} \cdot \left[\frac{\eta^{20}}{\theta_1(\epsilon_{1,2}\pm 2u)\theta_1(\epsilon_{1,2})^3\theta_3(\epsilon_{1,2})\theta_2(\epsilon_{1,2}\pm u)\theta_4(\epsilon_{1,2}\pm u)}\right]_{\text{sym}} \cdot \left[\prod_{l=1}^{8}\frac{\theta_1(m_l\pm u)\theta_2(m_l)\theta_4(m_l)}{\eta^4}\right]_{\text{Fermi}} \tag{3.48}$$

where the holonomy $(a_1, a_2, a_3, a_4) = (u, -u, 0, \frac{1+\tau}{2})$ and $(u, -u, \frac{\tau}{2}, \frac{1}{2})$ are used for $Z_{4(3)}$ and $Z_{4(3)'}$, respectively. Finally, the indices from the two sectors in the (eo) part are

$$Z_{4(4)} = \oint \left[\eta^2 du \cdot \frac{\theta_1(2\epsilon_+)\theta_4(2\epsilon_+)\theta_1(2\epsilon_+ \pm u)\theta_4(2\epsilon_+ \pm u)\theta_4(0)\theta_1(\pm u)\theta_4(\pm u)}{i\eta^{11}}\right]_{\text{vec}} \cdot \left[\frac{\eta^{20}}{\theta_1(\epsilon_{1,2}\pm 2u)\theta_1(\epsilon_{1,2})^3\theta_4(\epsilon_{1,2})\theta_1(\epsilon_{1,2}\pm u)\theta_4(\epsilon_{1,2}\pm u)}\right]_{\text{sym}} \cdot \left[\prod_{l=1}^{8}\frac{\theta_1(m_l\pm u)\theta_1(m_l)\theta_4(m_l)}{\eta^4}\right]_{\text{Fermi}} \tag{3.49}$$

$$Z_{4(4)'} = \oint \left[\eta^2 du \cdot \frac{\theta_1(2\epsilon_+)\theta_4(2\epsilon_+)\theta_2(2\epsilon_+ \pm u)\theta_3(2\epsilon_+ \pm u)\theta_4(0)\theta_2(\pm u)\theta_3(\pm u)}{i\eta^{11}}\right]_{\text{vec}} \cdot \left[\frac{\eta^{20}}{\theta_1(\epsilon_{1,2}\pm 2u)\theta_1(\epsilon_{1,2})^3\theta_4(\epsilon_{1,2})\theta_2(\epsilon_{1,2}\pm u)\theta_3(\epsilon_{1,2}\pm u)}\right]_{\text{sym}} \cdot \left[\prod_{l=1}^{8}\frac{\theta_1(m_l\pm u)\theta_2(m_l)\theta_3(m_l)}{\eta^4}\right]_{\text{Fermi}} \tag{3.50}$$

where the holonomy $(a_1, a_2, a_3, a_4) = (u, -u, 0, \frac{\tau}{2})$ and $(u, -u, \frac{1}{2}, \frac{1+\tau}{2})$ are used for $Z_{4(4)}$ and $Z_{4(4)'}$, respectively.

We also need to specify the residues which contribute to the above contour integrals. For the rank 1 cases, one just keeps all poles and residues associated with positively charged chiral fields. So for $Z_{4(i)}$ with $i = 2, 3, 4$, the relevant poles are at $u_* = -\frac{\epsilon_{1,2}}{2} + \frac{p}{2}$, where p runs over $(p_1, p_2, p_3, p_4) = (0, 1, 1+\tau, \tau)$, and $u_* = -\epsilon_{1,2},\ -\epsilon_{1,2} + \frac{p_i}{2}$. For $Z_{4(i)'}$ with $i = 2, 3, 4$, the poles are at $u_* = -\frac{\epsilon_{1,2}}{2} + \frac{p}{2}$, again with p running over $(p_1, p_2, p_3, p_4) = (0, 1, 1+\tau, \tau)$, and at $u_* = -\epsilon_{1,2} + p_j$ with two possible values of $j \neq 1, i$. The resulting residue sums are given by

$$\begin{aligned} Z_{4(2)} = {}& \frac{1}{2}\sum_{i=1}^{4} \frac{\theta_2(\epsilon_1+\epsilon_2)\theta_i(\frac{3\epsilon_1}{2}+\epsilon_2)\theta_{\sigma_2(i)}(\frac{3\epsilon_1}{2}+\epsilon_2)\theta_2(0)\theta_i(-\frac{\epsilon_1}{2})\theta_{\sigma_2(i)}(-\frac{\epsilon_1}{2})\prod_l\theta_1(m_l)\theta_2(m_l)\theta_i(m_l\pm\frac{\epsilon_1}{2})}{\eta^{24}\theta_1(2\epsilon_1)\theta_1(\epsilon_2-\epsilon_1)\theta_1(\epsilon_{1,2})^3\theta_2(\epsilon_{1,2})\theta_i(\frac{3\epsilon_1}{2})\theta_i(\epsilon_2-\frac{\epsilon_1}{2})\theta_{\sigma_2(i)}(\frac{3\epsilon_1}{2})\theta_{\sigma_2(i)}(\epsilon_2-\frac{\epsilon_1}{2})} \\ &+ \frac{\theta_2(2\epsilon_1+\epsilon_2)\theta_2(\epsilon_1)\left(\prod_l\theta_1(m_l\pm\epsilon_1)+\prod_l\theta_2(m_l\pm\epsilon_1)\right)\prod_l\theta_1(m_l)\theta_2(m_l)}{\eta^{24}\theta_1(3\epsilon_1)\theta_1(\epsilon_2-2\epsilon_1)\theta_1(\epsilon_{1,2})^2\theta_1(2\epsilon_1)\theta_1(\epsilon_2-\epsilon_1)\theta_2(2\epsilon_1)\theta_2(\epsilon_2-\epsilon_1)} + (\epsilon_1\leftrightarrow\epsilon_2) \end{aligned} \tag{3.51}$$

$$\begin{aligned} Z_{4(2)'} = {}& \frac{1}{2}\sum_{i=1}^{4} \frac{\theta_2(\epsilon_1+\epsilon_2)\theta_{\sigma_3(i)}(\frac{3\epsilon_1}{2}+\epsilon_2)\theta_{\sigma_4(i)}(\frac{3\epsilon_1}{2}+\epsilon_2)\theta_2(0)\theta_{\sigma_3(i)}(-\frac{\epsilon_1}{2})\theta_{\sigma_4(i)}(-\frac{\epsilon_1}{2})\prod_l\theta_3(m_l)\theta_4(m_l)\theta_i(m_l\pm\frac{\epsilon_1}{2})}{\eta^{24}\theta_1(2\epsilon_1)\theta_1(\epsilon_2-\epsilon_1)\theta_1(\epsilon_{1,2})^3\theta_2(\epsilon_{1,2})\theta_{\sigma_3(i)}(\frac{3\epsilon_1}{2})\theta_{\sigma_3(i)}(\epsilon_2-\frac{\epsilon_1}{2})\theta_{\sigma_4(i)}(\frac{3\epsilon_1}{2})\theta_{\sigma_4(i)}(\epsilon_2-\frac{\epsilon_1}{2})} \\ &+ \frac{\theta_2(2\epsilon_1+\epsilon_2)\theta_2(\epsilon_1)\left(\prod_l\theta_3(m_l\pm\epsilon_1)+\prod_l\theta_4(m_l\pm\epsilon_1)\right)\prod_l\theta_3(m_l)\theta_4(m_l)}{\eta^{24}\theta_1(3\epsilon_1)\theta_1(\epsilon_2-2\epsilon_1)\theta_1(\epsilon_{1,2})^2\theta_1(2\epsilon_1)\theta_1(\epsilon_2-\epsilon_1)\theta_2(2\epsilon_1)\theta_2(\epsilon_2-\epsilon_1)} + (\epsilon_1\leftrightarrow\epsilon_2) \end{aligned} \tag{3.52}$$

where σ_i are defined as (3.38). The expressions for $Z_{4(i)}$ and $Z_{4(i)'}$ with $i = 3, 4$ are obtained by permuting the roles of the subscripts 2, 3, 4 of the theta functions and σ_i.

The rank 2 contour integral in $Z_{4(1)}$ can be done as follows. The charges of the $(0,2)$ chiral multiplets, responsible for the poles in the integrand, are $\pm 2e_I$, $\pm e_I \pm e_J$ $(I \neq J)$ with $I, J = 1, 2$. We choose the vector η to be in the cone between $e_1 + e_2$ and $2e_2$. Then, the poles with nonzero Jeffrey-Kirwan residues (after eliminating the fake poles due to vanishing numerators from Fermi multiplets) are at the following 104 positions:

$$\begin{aligned}
&(1): && 2u_2+\epsilon=0,\ u_1+u_2+\epsilon'=0 &&\to u_2=-\frac{\epsilon}{2}+\frac{p_i}{2}, && u_1=-\epsilon'+\frac{\epsilon}{2}+\frac{p_i}{2}\\
&(2): && 2u_2+\epsilon=0,\quad 2u_1+\epsilon=0 &&\to u_2=-\frac{\epsilon}{2}+\frac{p_i}{2}, && u_1=-\frac{\epsilon}{2}+\frac{p_j}{2}\quad (p_i\neq p_j)\\
&(3): && 2u_2+\epsilon=0,\quad 2u_1+\epsilon'=0 &&\to u_2=-\frac{\epsilon}{2}+\frac{p_i}{2}, && u_1=-\frac{\epsilon'}{2}+\frac{p_j}{2}\\
&(4): && 2u_2+\epsilon=0,\ u_1-u_2+\epsilon=0 &&\to u_2=-\frac{\epsilon}{2}+\frac{p_i}{2}, && u_1=-\frac{3\epsilon}{2}+\frac{p_i}{2}\\
&(5): && u_2-u_1+\epsilon=0,\ u_1+u_2+\epsilon=0 &&\to u_2=-\epsilon+\frac{p_i}{2}, && u_1=0+\frac{p_i}{2}\\
&(6): && u_2-u_1+\epsilon=0,\ u_1+u_2+\epsilon'=0 &&\to u_2=-\frac{\epsilon+\epsilon'}{2}+\frac{p_i}{2}, && u_1=-\frac{\epsilon'-\epsilon}{2}+\frac{p_i}{2}\\
&(7): && u_2-u_1+\epsilon=0,\quad 2u_1+\epsilon=0 &&\to u_2=-\frac{3\epsilon}{2}+\frac{p_i}{2}, && u_1=-\frac{\epsilon}{2}+\frac{p_i}{2}\\
&(8): && -2u_1+\epsilon=0,\ u_1+u_2+\epsilon=0 &&\to u_1=+\frac{\epsilon}{2}+\frac{p_i}{2}, && u_2=-\frac{3\epsilon}{2}+\frac{p_i}{2}\,.
\end{aligned}\tag{3.53}$$

We defined $(p_1, p_2, p_3, p_4) = (0, 1, 1+\tau, \tau)$. ϵ can be either ϵ_1 or ϵ_2, and $\epsilon' \neq \epsilon$ is chosen between ϵ_1, ϵ_2 at given ϵ. In the second case, the four cases with $p_i = p_j$ do not provide poles since there are vanishing factors in the numerator. One can check that these poles are all non-degenerate.

The residue sums from these 8 cases are given by (the sectors labeled by (4), (7), (8) yield same result, shown on the second line)

$$\begin{aligned}
&(1): \sum_{i=1}^{4} \frac{\theta_1(2\epsilon_1+\epsilon_2)\theta_1(-\epsilon_1)\prod_l \theta_i(m_l \pm (\epsilon_1-\frac{\epsilon_2}{2}))\theta_i(m_l \pm \frac{\epsilon_2}{2})}{2\eta^{24}\theta_1(\epsilon_{1,2})^2\theta_1(2\epsilon_1)\theta_1(\epsilon_2-\epsilon_1)\theta_1(2\epsilon_1-\epsilon_2)\theta_1(2\epsilon_2-\epsilon_1)\theta_1(3\epsilon_1-\epsilon_2)\theta_1(2\epsilon_2-2\epsilon_1)} + (\epsilon_1 \leftrightarrow \epsilon_2)\\
&(4): \sum_{i=1}^{4} \frac{\prod_l \theta_i(m_l \pm \frac{\epsilon_1}{2})\theta_i(m_l \pm \frac{3\epsilon_1}{2})}{2\eta^{24}\theta_1(\epsilon_{1,2})\theta_1(2\epsilon_1)\theta_1(3\epsilon_1)\theta_1(4\epsilon_1)\theta_1(\epsilon_2-\epsilon_1)\theta_1(\epsilon_2-2\epsilon_1)\theta_1(\epsilon_2-3\epsilon_1)} + (\epsilon_1 \leftrightarrow \epsilon_2) = (7) = (8)\\
&(5): \sum_{i=1}^{4} \frac{\theta_1(2\epsilon_1+\epsilon_2)\theta_1(-\epsilon_1)\prod_l \theta_i(m_l)^2\theta_i(m_l \pm \epsilon_1)}{2\eta^{24}\theta_1(\epsilon_{1,2})^2\theta_1(2\epsilon_1)^2\theta_1(\epsilon_2-\epsilon_1)^2\theta_1(3\epsilon_1)\theta_1(\epsilon_2-2\epsilon_1)} + (\epsilon_1 \leftrightarrow \epsilon_2)\\
&(6): \sum_{i=1}^{4} \frac{\prod_l \theta_i(m_l \pm \frac{\epsilon_1+\epsilon_2}{2})\theta_i(m_l \pm \frac{\epsilon_1-\epsilon_2}{2})}{\eta^{24}\theta_1(\epsilon_{1,2})\theta_1(2\epsilon_1)\theta_1(\epsilon_1-\epsilon_2)\theta_1(2\epsilon_2)\theta_1(\epsilon_2-\epsilon_1)\theta_1(2\epsilon_1-\epsilon_2)\theta_1(2\epsilon_2-\epsilon_1)}
\end{aligned}\tag{3.54}$$

and

$$\begin{aligned}
(2): \Bigg[&\frac{\theta_2(0)\theta_2(-\epsilon_1)\theta_2(\epsilon_1+\epsilon_2)\theta_2(2\epsilon_1+\epsilon_2)\left(\prod_l \theta_1(m_l \pm \frac{\epsilon_1}{2})\theta_2(m_l \pm \frac{\epsilon_1}{2}) + \prod_l \theta_3(m_l \pm \frac{\epsilon_1}{2})\theta_4(m_l \pm \frac{\epsilon_1}{2})\right)}{2\eta^{24}\theta_1(\epsilon_{1,2})^2\theta_1(2\epsilon_1)^2\theta_1(\epsilon_2-\epsilon_1)^2\theta_2(\epsilon_{1,2})\theta_2(2\epsilon_1)\theta_2(\epsilon_2-\epsilon_1)}\\
&+ (2,3,4 \to 3,4,2) + (2,3,4 \to 4,2,3)\Bigg] + (\epsilon_1 \leftrightarrow \epsilon_2)
\end{aligned}\tag{3.55}$$

$$(3): \sum_{i,j=1}^{4} \frac{\prod_l \theta_j(m_l \pm \frac{\epsilon_1}{2})\theta_i(m_l \pm \frac{\epsilon_2}{2})}{2\eta^{24}\theta_1(\epsilon_{1,2})^2\theta_1(2\epsilon_1)\theta_1(\epsilon_2-\epsilon_1)\theta_1(2\epsilon_2)\theta_1(\epsilon_1-\epsilon_2)}\ \frac{\theta_{\sigma_j(i)}(-\frac{\epsilon_1+\epsilon_2}{2})\theta_{\sigma_j(i)}(\frac{3(\epsilon_1+\epsilon_2)}{2})}{\theta_{\sigma_j(i)}(\frac{3\epsilon_1-\epsilon_2}{2})\theta_{\sigma_j(i)}(\frac{3\epsilon_2-\epsilon_1}{2})}\,. \tag{3.56}$$

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$Z_{4(1)}$ is given by the sum of eight contributions $(1), \cdots, (8)$. The full index is given by

$$Z_4 = \frac{1}{8}\sum_{i=1}^{4} Z_{4(i)} + \frac{1}{8}\sum_{i=2}^{4} Z_{4(i)'} + \frac{1}{16} Z_{4(1)'}\,, \tag{3.57}$$

with the Weyl factors given by (3.9).

We test our results against various known ones. We first consider the case in which one sets

$$\epsilon_1 = -\epsilon_2 \equiv \epsilon,\ m_1 = m_2 = 0, m_3 = m_4 = \frac{1}{2},\ m_5 = m_6 = -\frac{1+\tau}{2},\ m_7 = m_8 = \frac{\tau}{2}\,. \tag{3.58}$$

This case was considered recently in [33]. In particular, [33] wrote down the concrete forms of the elliptic genera in this limit for 2 and 4 E-strings. The case with 2 E-strings is a special case of [7], so also agrees with our results. The index of [33] at (3.58) is always zero for odd number of E-strings. By plugging in (3.58) to our 3 E-string indices in the previous subsection, all $Z_{3(i)}, Z_{3(i)'}$ are identically zero, agreeing with the results of [33]. Now let us study our 4 E-string index. Plugging in (3.58), one finds that the contributions from the seven sectors are zero, and the only nonzero contribution is $Z_{4(1)}$. The surviving contributions are

$$\begin{aligned}(1) = (4) = (7) = (8) &= \frac{4\prod_{i=1}^{4}\theta_i(3\epsilon/2)^4\theta_i(\epsilon/2)^4}{\eta^{24}\theta_1(\epsilon)^2\theta_1(2\epsilon)^2\theta_1(3\epsilon)^2\theta_1(4\epsilon)^2} \\ (2) = (3) &= \frac{2\prod_i\theta_i(\epsilon/2)^8}{\eta^{24}\theta_1(\epsilon)^4\theta_1(2\epsilon)^4}\left[\frac{\theta_2(0)^2}{\theta_2(2\epsilon)^2} + \frac{\theta_3(0)^2}{\theta_3(2\epsilon)^2} + \frac{\theta_4(0)^2}{\theta_4(2\epsilon)^2}\right]\end{aligned} \tag{3.59}$$

while $(5), (6)$ become zero. So one obtains

$$\begin{aligned}Z_{4(1)} &= \frac{16\prod_{i=1}^{4}\theta_i(\frac{3\epsilon}{2})^4\theta_i(\frac{\epsilon}{2})^4}{\eta^{24}\theta_1(\epsilon)^2\theta_1(2\epsilon)^2\theta_1(3\epsilon)^2\theta_1(4\epsilon)^2} + \frac{4\prod_i\theta_i(\frac{\epsilon}{2})^8}{\eta^{24}\theta_1(\epsilon)^4\theta_1(2\epsilon)^4}\left[\frac{\theta_2(0)^2}{\theta_2(2\epsilon)^2} + \frac{\theta_3(0)^2}{\theta_3(2\epsilon)^2} + \frac{\theta_4(0)^2}{\theta_4(2\epsilon)^2}\right] \\ &= \frac{16\theta_1(\epsilon)^2\theta_1(3\epsilon)^2}{\theta_1(2\epsilon)^2\theta_1(4\epsilon)^2} + \frac{4\theta_1(\epsilon)^4}{\theta_1(2\epsilon)^4}\left[\frac{\theta_2(0)^2}{\theta_2(2\epsilon)^2} + \frac{\theta_3(0)^2}{\theta_3(2\epsilon)^2} + \frac{\theta_4(0)^2}{\theta_4(2\epsilon)^2}\right].\end{aligned} \tag{3.60}$$

The four E-string index at (3.58) is given in [33] by

$$\frac{\theta_1(\epsilon)^{20}}{2\eta^{48}\theta_1(2\epsilon)^2\theta_1(4\epsilon)^2}\left[72(\wp')^4\wp^2 - 18(\wp'')^2(\wp')^2\wp + 2\wp''(\wp')^4 + (\wp'')^4\right], \tag{3.61}$$

where $\wp(\tau, \epsilon)$ is the Weierstrass's elliptic function. We checked that this agrees with our index $\frac{1}{8}Z_{4(1)}$ in a serious expansion in q for the first 11 terms, up to and including $\mathcal{O}(q^{10})$.

We also compare our result with the genus expansion, at $m_l = 0$ and $\epsilon_1 = -\epsilon_2 = \epsilon$. Our indices become

$$Z_{4(1)} = \sum_{i=1}^{4} \left[\frac{4\theta_i(\frac{3\epsilon}{2})^{16}\theta_i(\frac{\epsilon}{2})^{16}}{\eta^{24}\theta_1(\epsilon)^2\theta_1(2\epsilon)^2\theta_1(3\epsilon)^2\theta_1(4\epsilon)^2} + \frac{2\theta_i(0)^{16}\theta_i(\epsilon)^{16}}{\eta^{24}\theta_1(\epsilon)^2\theta_1(2\epsilon)^4\theta_1(3\epsilon)^2} \right] \tag{3.62}$$
$$+ \frac{2}{\eta^{24}\theta_1(\epsilon)^4\theta_1(2\epsilon)^4} \left[\frac{\theta_2(0)^2(\theta_1(\frac{\epsilon}{2})^{16}\theta_2(\frac{\epsilon}{2})^{16} + \theta_3(\frac{\epsilon}{2})^{16}\theta_4(\frac{\epsilon}{2})^{16})}{\theta_2(2\epsilon)^2} + (3,4,2) + (4,2,3) \right]$$
$$Z_{4(2)'} = \sum_{i=1}^{4} \frac{\theta_2(0)^2\theta_{\sigma_3(i)}(\frac{\epsilon}{2})^2\theta_{\sigma_4(i)}(\frac{\epsilon}{2})^2\theta_3(0)^8\theta_4(0)^8\theta_i(\frac{\epsilon}{2})^{16}}{\eta^{24}\theta_1(2\epsilon)^2\theta_1(\epsilon)^6\theta_2(\epsilon)^2\theta_{\sigma_3(i)}(\frac{3\epsilon}{2})^2\theta_{\sigma_4(i)}(\frac{3\epsilon}{2})^2} + \frac{2\theta_2(\epsilon)^2\theta_3(0)^8\theta_4(0)^8(\theta_3(\epsilon)^{16} + \theta_4(\epsilon)^{16})}{\eta^{24}\theta_1(3\epsilon)^2\theta_1(2\epsilon)^2\theta_1(\epsilon)^4\theta_2(2\epsilon)^2},$$

with $Z_{4(1)'} = 0$, $Z_{4(2)} = Z_{4(3)} = Z_{4(4)} = 0$, and $Z_{4(3)'}$, $Z_{4(4)'}$ are obtained from $Z_{4(2)'}$ by changing the roles of $2, 3, 4$ appearing in the subscripts of the theta functions and $\sigma_2(i), \sigma_3(i), \sigma_4(i)$. We first confirmed numerically the agreement with $F^{(0,g,4)}$ computed from topological strings for $g \leq 5$ till q^5, by checking the first 10 terms in the serious expansion in q. We also exactly checked the agreements of $F^{(0,0,4)}$, $F^{(0,1,4)}$, $F^{(0,2,4)}$. See appendix C for the details.

3.5 Higher E-strings

The computation of the elliptic genus using the methods of [24] quickly becomes complicated for higher rank gauge groups. In general, there could be a fundamental complication due to some poles failing to be projective. But we showed at the beginning of this section that this does not happen in our problem. So the computation of the elliptic genus can be done using our methods for any number of E-strings. With higher rank, the computational problem is that there is a large number of poles and residues to be considered. For U(n) indices, the possible poles are often completely classified by the so-called 'colored Young diagrams.' This classification first appeared in the context of instanton counting [44, 45], which was reproduced recently in the context of Jeffrey-Kirwan residues [34]. The resulting residues are often nicely arranged into a reasonably compact form [46, 47]. However, for other gauge groups, we are not aware of systematic classifications of poles.[3] In this subsection, we shall illustrate the pole structures for some higher E-strings, with $O(5)$, $O(6)$, $O(7)$, $O(8)$ gauge groups, and also make some qualitative classifications of these poles. Since the purpose is to illustrate the computations for higher ranks, we only consider the branch of $O(n)$ holonomy with maximal number of continuous parameters, in the (ee) sector.

We start by studying the $O(5)$ index, for five E-strings. Taking $\eta = e_1 + \varepsilon e_2$ with $0 < \varepsilon \ll 1$, the following pair of weights $\{\rho_1, \rho_2\}$ can potentially give nonzero JK-Res:

$$\{2e_1, 2e_2\},\ \{2e_1, e_2\},\ \{2e_1, e_2 \pm e_1\},\ \{e_1, 2e_2\},\ \{e_1, e_2\},\ \{e_1, e_2 \pm e_1\} \tag{3.63}$$
$$\{e_1 - e_2, 2e_2\},\ \{e_1 - e_2, e_1 + e_2\},\ \{e_1 - e_2, e_2\},\ \{e_1 + e_2, -2e_2\},\ \{e_1 + e_2, -e_2\}.$$

[3] The pole structure of our $O(n)$ index is similar to that of the Sp(N) instanton partition function, whose ADHM quantum mechanics comes with $O(n)$ group for n instantons. The poles in our E-string index could be slightly simpler, because we only have $O(n)$ symmetric hypermultiplets while the ADHM mechanics also has extra N fundamental hypermultiplets. In either case, we do not know the pole classification, apart from the basic rule given by the Jeffrey-Kirwan residues.

These poles define the pole u_* by hyperplanes $\rho_i(u_*)+z_i=0$ for suitable z_i, chosen between ϵ_1, ϵ_2. Considering all possible values of u_*, we find 142 poles, which are all non-degenerate. The evaluation of residue sum should be marginally more laborious than the $O(4)$ case.

Next, we consider the $O(6)$ contour integral. The poles come from the scalar fields with charges $\pm 2e_I$, $\pm e_I \pm e_J$. We choose η to be $\eta = e_1 + \varepsilon e_2 + \varepsilon^2 e_3$ with $0 < \varepsilon \ll 1$. The groups of 3 vectors which contain η in their cones are

$$\begin{aligned}
&\{2e_1, 2e_2, 2e_3\},\ \{2e_1, 2e_2, e_3 \pm e_{1,2}\},\ \{2e_1, 2e_3, e_2 \pm e_1\},\ \{2e_1, 2e_3, e_2 - e_3\},\ \{2e_1, -2e_3, e_2 + e_3\},\\
&\{2e_1, e_2 \pm e_1, e_3 \pm e_1\},\ \{2e_1, e_2 \pm e_1, e_3 \pm e_2\},\ \{2e_1, e_3 \pm e_1, e_2 - e_3\},\ \{2e_1, -e_3 \pm e_1, e_2 + e_3\}\\
&\{2e_1, e_2 + e_3, e_2 - e_3\},\ \{2e_2, 2e_3, e_1 - e_{2,3}\},\ \{2e_2, -2e_3, e_1 + e_3\},\ \{2e_2, e_1 - e_2, e_3 \pm e_{1,2}\}\\
&\{2e_2, e_1 + e_3, e_1 - e_3\},\ \{2e_2, e_1 + e_3, -e_2 - e_3\},\ \{2e_2, e_1 - e_3, -e_2 + e_3\},\ \{2e_3, -2e_2, e_1 + e_2\},\\
&\{2e_3, e_1 + e_2, e_1 - e_{2,3}\},\ \{2e_3, e_1 + e_2, -e_2 - e_3\},\ \{2e_3, e_1 - e_2, e_2 - e_3\},\ \{2e_3, e_2 - e_1, e_1 - e_3\},\\
&\{2e_3, e_1 - e_3, e_2 \pm e_3\},\ \{-2e_2, e_1 + e_2, e_3 \pm e_{1,2}\},\ \{-2e_2, e_1 + e_3, e_2 - e_3\},\ \{-2e_2, e_1 - e_3, e_2 + e_3\},\\
&\{-2e_3, e_1 + e_2, e_1 + e_3\},\ \{-2e_3, e_1 + e_2, -e_2 + e_3\},\ \{-2e_3, e_1 - e_2, e_2 + e_3\},\ \{-2e_3, e_2 - e_1, e_1 + e_3\},\\
&\{-2e_3, e_1 + e_3, e_2 \pm e_3\},\ \{e_1 + e_2, e_1 - e_2, e_3 \pm e_{1,2}\},\ \{e_1 + e_2, e_1 + e_3, e_1 - e_3\},\\
&\{e_1 + e_2, e_1 + e_3, -e_2 - e_3\},\ \{e_1 + e_2, e_1 - e_3, -e_2 + e_3\},\ \{e_1 + e_2, e_3 - e_2, -e_2 - e_3\},\\
&\{e_1 - e_2, e_1 + e_3, e_2 - e_3\},\ \{e_1 - e_2, e_1 - e_3, e_2 + e_3\}, \{e_1 - e_2, e_2 + e_3, e_2 - e_3\},\\
&\{e_2 - e_1, e_1 + e_3, e_1 - e_3\},\ \{e_1 + e_3, e_1 - e_3, e_2 \pm e_3\},\ \{e_1 + e_3, e_2 - e_3, -e_2 - e_3\},\\
&\{e_1 - e_3, e_2 + e_3, e_3 - e_2\}\,.
\end{aligned} \tag{3.64}$$

With these chosen $\{\rho_1, \rho_2, \rho_3\}$, the hyperplanes $\rho_i(u_*) + z_i = 0$ with $i = 1, 2, 3$ meet at a point u_* with suitable choices of z_i, which are either ϵ_1 or ϵ_2. There may exist more than the chosen three hyperplanes which meet at the same point u_*, in which case we have degenerate poles. Also, at some u_* there could be some vanishing theta functions in the numerator. Let us call the number of vanishing theta functions from the numerator and denominator as $N_n(u_*)$ and $N_d(u_*)$, respectively. When $N_d - N_u < r = 3$, then the corresponding u_* is not a pole due to too many vanishing terms in the numerator. The list below covers all the poles which have nonzero JK-Res, also provided with some illustrations on how to evaluate the residues:

1. When $N_d = 3$, $N_n = 0$, this is a non-degenerate and simple pole. We find 1680 poles in this class. Near $u = u_*$, the integrand relevant for evaluating the residue approximately takes the form of

$$\frac{1}{\prod_{i=1}^{r}(\rho_i(u) - \rho(u_*))} \cdot F(u_*)\,, \tag{3.65}$$

   where $F(u)$ denotes the rest of the integrand, with $F(u_*) \neq 0$. The integral of the first factor of (3.65) can be immediately obtained from the basic definition (3.17).

2. There could be degenerate poles with $N_d = N_n + r$, $N_n \neq 0$. The leading divergences of the integrands are simple poles in this case, since $N_d - N_n = r$. Near the pole, the integrand relevant for computing the residue approximately takes the form of

$$\frac{\prod_{i=1}^{N_n}(\rho_i(u) - \rho_i(u_*))}{\prod_{i=N_n+1}^{r+2N_n}(\rho_i(u) - \rho_i(u_*))} \cdot F(u_*)\,, \tag{3.66}$$

where $F(u)$ is the rest of the integrand. The basic rule (3.17) has to be applied to the first factor of (3.66) after decomposing it into a linear combination of the expressions appearing in (3.17). In the $O(6)$ case with $r = 3$, we find two subclasses. Firstly, we find 104 poles with $N_d = 4$, $N_n = 1$. For all the poles in this class, we find

$$\text{JK-Res}\frac{\rho_1(u) - \rho_1(u_*)}{\prod_{i=2}^5(\rho_i(u) - \rho_i(u_*))} = \frac{1}{2}\,, \tag{3.67}$$

thus all with nonzero residues. We illustrate how this is evaluated with an example among the 104 poles, defined with $\{\rho_1, \rho_2, \rho_3, \rho_4\} = \{e_1 - e_2, e_1 + e_2, e_1 + e_3, -e_2 - e_3, -2e_2\}$:

$$\begin{aligned} &\text{JK-Res}\frac{\bigwedge_{a=1}^3 du_a \cdot (\epsilon_1 + \epsilon_2 + u_1 - u_2)}{(\epsilon_1 - 2u_2)(\epsilon_2 + u_1 + u_2)(\epsilon_2 - u_2 - u_3)(\epsilon_1 + u_2 + u_3)} \\ &= \text{JK-Res}\frac{\bigwedge_{a=1}^3 d\tilde{u}_a}{(\tilde{u}_1 + \tilde{u}_3)(-\tilde{u}_2 - \tilde{u}_3)}\left(\frac{1}{\tilde{u}_1 + \tilde{u}_2} + \frac{1}{-2\tilde{u}_2}\right) = \frac{1}{2} + 0 = \frac{1}{2}\,, \end{aligned} \tag{3.68}$$

where $\tilde{u} = u - u_*$. Moreover, we find 72 poles with $N_d = 5$, $N_n = 2$, in which case we find either

$$\begin{aligned} &\text{JK-Res}\frac{(\rho_1(u) - \rho_1(u_*))(\rho_2(u) - \rho_2(u_*))}{\prod_{i=3}^7(\rho_i(u) - \rho_i(u_*))} = \\ &0 \text{ (32 cases)},\ -\frac{1}{4} \text{ (16 cases)},\ \frac{1}{4},\ \text{(16 cases)}\ \frac{1}{2} \text{ (8 cases)}\,. \end{aligned} \tag{3.69}$$

Thus we find 40 more poles. There are no more poles in this class with larger N_d, N_n.

3. In general, there could be degenerate poles with $N_d > N_n + r$. The integrand contains 'multiple poles' in this case. The integrand takes the form of

$$\frac{\prod_{i=1}^{N_n}\theta_1(\rho_i(u) - \rho_i(u_*))}{\prod_{i=N_n+1}^{N_d+N_n}\theta_1(\rho_i(u) - \rho_i(u_*))} \cdot F(u)\,, \tag{3.70}$$

where $F(u)$ is a combination of θ_1 functions which are nonzero at u_*. Since the first factor contains multiple poles, one would have to expand both first and second factors to certain orders near $u = u_*$, until one obtains a linear combination of the functions appearing in (3.17). The residue will thus be expressed by θ_1 functions and their suitable derivatives at u_*. This class of poles do not show up in the $O(6)$ case. (They will first appear in the $O(8)$ index, explained below.)

With the above $1680 + 104 + 40 = 1824$ poles and the computational rules stated in the list, clearly the $O(6)$ elliptic genus can be computed straightforwardly, although the resulting expression will be very long.

Let us explain the pole/residue structures of $O(7)$ index, with rank $r = 3$. The poles are again classified into the above three classes. To be definite, we chose $\eta = e_1 + \varepsilon e_2 + \varepsilon^2 e_3$. We simply list the number poles in each class.

1. non-degenerate poles ($N_d = 3$, $N_n = 0$): 2468 cases

2. degenerate (but simple) poles: with $N_d = 4$, $N_n = 1$, we find 106 degenerate and simple poles. The relevant integrals of the form of (3.67) are either $\frac{1}{2}$ or 1, depending on u_*. With $N_d = 5$, $N_n = 2$, we find 72 cases. The integral analogous to (3.69) are either $0, -\frac{1}{4}, \frac{1}{4}, \frac{1}{2}$. There are 32 cases with zero residues. So we find 40 poles in this class. Finally, there are 4 cases with $N_d = 6$, $N_n = 3$, and the JK-Res of the rational functions are either

$$\text{JK-Res}\frac{\bigwedge_{a=1}^{r} d\tilde{u}_a \cdot \prod_{i=1}^{3}\rho_i(\tilde{u})}{\prod_{i=4}^{r+6}\rho_i(\tilde{u})} = \frac{1}{2} \quad (2 \text{ cases}), \text{ or } \quad 0 \quad (2 \text{ cases})\,. \tag{3.71}$$

So we have 2 poles in the last class. We do not find further degenerate simple poles with larger N_n.

3. degenerate multiple poles ($N_d > N_n + 3$): we do not find any poles in this case.

So we find $2468 + 106 + 40 + 2 = 2616$ poles with nonzero JK-Res.

As a final illustration, let us consider the $O(8)$ contour integral with rank $r = 4$. The number of poles quickly increases, as follows:

1. non-degenerate poles ($N_d = 4$, $N_n = 0$): 32304 poles

2. degenerate (but simple) poles: with $N_d = 5$, $N_n = 1$, we find 4424 poles. With $N_d = 6$, $N_n = 2$, we find 1696 poles. With $N_d = 7$, $N_n = 3$, we find 88 poles. Finally, with $N_d = 8$, $N_n = 4$, we finds 200 poles.

3. degenerate multiple poles ($N_d > N_n + 3$): we find 72 such poles.

So we find $32304 + 4424 + 1696 + 88 + 200 + 72 = 38784$ poles for the $O(8)$ contour integral.

4 E-strings from Yang-Mills instantons

In this section, we explain how one can alternatively compute the E-string elliptic genus from the instanton partition function of a suitable 5 dimensional super-Yang-Mills theory with Sp(1) gauge group. The basic idea is that suitable circle reductions of 6d SCFTs sometimes admit 5d SYM descriptions at low energy. The latter SYM, despite being non-renormalizable, remembers the 6d KK degrees in its solitonic sector as the instanton solitons [48, 49]. The self-dual strings wrapping the circle become the W-bosons, quarks or their superpartner particles in 5d. So the Witten index for the threshold bounds of these particles with instantons in the Coulomb branch [44, 45] will carry information on the elliptic genera of wrapped self-dual strings. This idea has been used to study the elliptic genus of M-strings in the 6d (2, 0) SCFT in [5, 8]. In this section, we make a similar study for the E-strings. Since the circle reduction of the E_8 (1, 0) SCFT is subtler than that of the (2, 0) theory, let us set up the problem first.

We start by considering the type IIA system consisting of 8 D8-branes and an O8-plane (or 16 D8-branes in the covering space), making a type I' string background. The D8-branes are at the tip of the half-line $\mathbb{R}^+$, formed by an O8. The worldvolume of the 8-branes hosts SO(16) gauge symmetry. Since the net 8-brane charges cancel, the asymptotic value of the dilaton on $\mathbb{R}^+$ is a nonzero constant. So this system admits an M-theory uplift at strong coupling, on $\mathbb{R}^{8+1} \times \mathbb{R}^+ \times S^1$. The D0-branes in the type I' theory are identified as the Kaluza-Klein modes along the M-theory circle. In the uplifted background, an M9-plane (or the Horava-Witten wall) is located at the tip of $\mathbb{R}^+$ and wraps $\mathbb{R}^{8+1} \times S^1$. The M9-plane hosts an E_8 gauge symmetry. When the M9 wraps a circle, one can turn on nonzero E_8 Wilson line which reduces gauge symmetry. To get a background which admits a weakly coupled type I' description with unbroken SO(16) gauge symmetry, one should turn on the Wilson line as follows. Let R be the radius of the M-theory circle, and A be the E_8 gauge field on the circle. E_8 has an SO(16) subgroup, in which the adjoint representation **248** of E_8 decomposes into $\mathbf{120} \oplus \mathbf{128}$. The Wilson line RA that we turn on in SO(16) $\subset E_8$ is given by [13]

$$RA = (0, 0, 0, 0, 0, 0, 0, 1)\,. \tag{4.1}$$

This is in the convention that one picks the Cartans of SO(16) as rotations on the 8 orthogonal 2-planes. The circle holonomy generated by this Wilson line is $\exp(2\pi i RA \cdot F)$, with $F = (F_1, F_2, \cdots, F_8)$ being the Cartans of SO(16) $\subset E_8$ in the same basis. The normalization is $F_l = \pm\frac{1}{2}$ for SO(16) spinors. The holonomy with (4.1) acts on **128** as -1, and on **120** as $+1$. So E_8 symmetry breaks down to SO(16). This is the background which admits the type I' theory description for small R.

Now let us consider the D4-D8-O8 system, by adding N D4-branes. This uplifts in M-theory to the M5-M9-branes wrapping the circle, in the above E_8 Wilson line background. On the worldvolume of D4-branes, one obtains an Sp(N) gauge theory with 1 antisymmetric and 8 fundamental hypermultiplets.[4] This 5d gauge theory is a low-energy description of the 6d (1,0) superconformal field theory compactified on a circle with E_8 Wilson line. Note that, from the worldvolume theory on D4 or M5-branes, SO(16) or E_8 act as global symmetries. So from the 5d/6d field theories, the Wilson line we explained above are nondynamical background fields.

Consider the system consisting of single M5-brane and an M9-plane, compactified on a circle with the above Wilson line. We have an Sp(1) gauge theory description in 5d. Taking into account the effect of the background Wilson line (4.1), we can identify various charges of the 5d SYM theory and the 6d (1,0) theory on circle as follows:

$$k = 2P + n(RA \cdot RA) - 2\left(RA \cdot \tilde{F}\right) = 2P + n - 2\tilde{F}_8 \tag{4.2}$$

$$F_l = \tilde{F}_l - n(RA_l) \quad \rightarrow \quad F_8 = \tilde{F}_8 - n\,. \tag{4.3}$$

Here, k, F_l appearing on the left hand sides are various charges of the 5d SYM, while $P, \tilde{F}_l$ on the right hand sides are those of the 6d E-string theory. k is the Yang-Mills instanton charge on D4's (i.e. D0-brane number in the type I' theory), P is the momentum on E-strings along the circle, $\tilde{F}_l$ are the E_8 Cartan charge in the 6d theory (which were called

[4] Had one been reducing the M5-M9 system with zero Wilson line, one would have obtained the strongly interacting 5d SCFT with E_8 symmetry [19, 50], discovered in [51].

F_l till here in this paper), and F are the SO(16) Cartan charges in the 5d SYM. n is the U(1) $\subset$ Sp(1) electric charge in the Coulomb phase, which is identified with the winding number of the E-strings. This formula can be naturally inferred by starting from the charge relations of the fundamental type I' stings on $\mathbb{R}^{8+1} \times I$ and the heterotic strings on $\mathbb{R}^{8+1} \times S^1$ [52, 53], where I is a segment, and then putting an M5-brane on I to decompose a heterotic string into two E-strings [7].

Later in this section, we shall consider an index for the E-strings, with the weight given by

$$q^k e^{2\pi i m_8 F_8} w^n \prod_{l=1}^{7} e^{2\pi i m_l F_l} = q^{2P} (y_8')^{\tilde{F}_8} (w')^n \prod_{l=1}^{7} e^{2\pi i m_l \tilde{F}_l} \tag{4.4}$$

with $y_i \equiv e^{2\pi i m_i}$, where

$$y_8' = y_8 q^{-2} \ , \quad w' = w q y_8^{-1} \, . \tag{4.5}$$

The right hand side of (4.4), with primes and tildes for fugacities and charges, is the natural expression for the E-strings from the 6d perspective, while the instanton calculus will naturally use the expression on the left hand side. After computing the instanton partition function with the above weight, we shall express it in terms of the fugacities y_8', w' given by (4.5), which can be compared with the E-string elliptic genus that we studied in this paper. This redefinition of fugacities plays the role of canceling the background E_8 Wilson line (4.1), which obscures the E_8 symmetry in the type I' instanton calculus.[5]

Since the ADHM quantum mechanics is a UV completion of the 5d instanton quantum mechanics, it contains extra string theory degrees of freedom apart from the QFT states. So the partition function of the ADHM quantum mechanics may acquire contributions from the extra string theory states in the D4-D8-O8 background. Since the 5d/6d quantum field theories are obtained from the string theory background by taking low energy decoupling limit, the Hilbert space of this system factorizes at low energy. In particular, in the context of the Witten index of the ADHM quantum mechanics, one expects

$$Z_{\text{ADHM}} = Z_{\text{inst}} \cdot Z_{\text{other}} \, . \tag{4.6}$$

The quantity of our interest is the 5d instanton partition function Z_{inst}. The factor Z_{other} was identified in [34]. For the purpose of studying the QFT spectrum, we simply divide the ADHM quantum mechanics partition function by Z_{other} identified in [34], to obtain Z_{inst}. See section 3.4.2 of [34] for the details.

We will consider the QFT partition function $Z_{\text{QFT}}(q, w, , m_l, \epsilon_{1,2})$ of the 5d Sp(1) gauge theory, i.e., the rank 1 6d (1, 0) SCFT compactified on circle with E_8 Wilson line. The full partition function is obtained by multiplying the 5d perturbative part Z_{pert} to Z_{inst}, i.e.

$$Z_{\text{QFT}}(q, w, m_l, \epsilon_{1,2}) = Z_{\text{pert}}(w, m_l, \epsilon_{1,2}) Z_{\text{inst}}(q, w, m_l, \epsilon_{1,2}) \, , \tag{4.7}$$

[5] Only in this section, the definition of q is given by $q = e^{\pi i \tau}$, instead of $q = e^{2\pi i \tau}$ used in all other sections of this paper. This is because the single instanton carries $q^{\frac{1}{2}}$ factor in the other convention, due to the fractional Wilson line, which we want to change to q^1. This is the reason for the factor q^{2P} in (4.4).

with

$$Z_{\rm pert} \equiv \exp\left[\sum_{n=1}^{\infty}\frac{1}{n}f(w^n, nm_l, n\epsilon_{1,2})\right], \quad f(w,m_l,\epsilon_{1,2}) \equiv \frac{\chi^{\rm SO(16)}_{\bf 16}(m_l)w - 2\cos(2\pi\epsilon_+)w^2}{(2i\sin\pi\epsilon_1)(2i\sin\pi\epsilon_2)}\,. \tag{4.8}$$

The first term of f comes from the quarks of the $N_f = 8$ Sp(1) fundamental hypermultiplets, where $\chi^{\rm SO(16)}_{\bf 16} \equiv \sum_{l=1}^{8}(e^{2\pi i m_l} + e^{-2\pi i m_l})$ is the character of **16**. The second term of f comes from the Sp(1) W-boson and superpartners in the vector multiplet. To study $Z_{\rm QFT}$ from the 6d E-string perspective, one first considers the grand partition function of the E-string elliptic genera $Z_n(q, m'_l, \epsilon_{1,2})$ that we studied in this paper,

$$Z_{\text{E-string}}(w', m'_l, \epsilon_{1,2}) = \sum_{n=0}^{\infty}(w')^n Z_n(q, m'_l, \epsilon_{1,2})\,, \tag{4.9}$$

where $Z_0 \equiv 1$. This captures the contribution to partition function $Z_{\rm QFT}$ from the states with nonzero E-string winding number n. One has to multiply the contribution from states at zero winding. For the E-string theory in the Coulomb branch, it comes from an $\mathcal{N} = (1,0)$ tensor multiplet, which is

$$Z_{\rm tensor}(q,\epsilon_{1,2}) \equiv \exp\left[\sum_{n=1}^{\infty}\frac{1}{n}g(q^n, n\epsilon_{1,2})\right], \quad g(q,\epsilon_{1,2}) \equiv -\frac{2\cos(2\pi\epsilon_-)}{(2i\sin\pi\epsilon_1)(2i\sin\pi\epsilon_2)}\frac{q^2}{1-q^2}\,. \tag{4.10}$$

g is the single particle index of a $(1,0)$ tensor multiplet on a circle [8].[6] Then, one finds

$$Z_{\rm QFT}(q, w, m_l, \epsilon_{1,2}) = Z_{\text{E-string}}(w', m'_l, \epsilon_{1,2}) Z_{\rm tensor}(q, \epsilon_{1,2})\,. \tag{4.11}$$

With (4.5), this provides the second formula for $Z_{\rm QFT}$. The expression (4.7) takes the form of series expansion in q, since we know the coefficients of $Z_{\rm inst}(q, w, m_l, \epsilon_{1,2}) = \sum_{k=0}^{\infty} Z_k(w, m_l, \epsilon_{1,2})q^k$. So at a given order in the modular parameter q, one captures the spectrum of arbitrary number of E-strings by computing Z_k exactly in w. This is in contrast to the formula (4.11) obtained from the E-string elliptic genus, keeping definite order $Z_n(q, m'_l, \epsilon_{1,2})$ in $w'(\sim w)$ which is exact in q. So to confirm that the two approaches yield the same result, we shall make a double expansions of (4.7) and (4.11) in q, w and compare, taking into account the shifts (4.5). While making the study of instanton partition function of our Sp(1) gauge theory in [34], $Z_k(w, m_l, \epsilon_{1,2})$ was computed up to $k = 5$. So expanding $Z_n(q, y'_8, \epsilon_{1,2}) = Z_n(q, y_8 q^{-2}, \epsilon_{1,2})$ up to $\mathcal{O}(q^5)$ at fixed $y_8 = e^{2\pi i m_8}$, and expanding $Z_{\rm QFT}$ computed from 5d to $\mathcal{O}(w^n)$ for some low n, we shall find perfect agreement of the two results.

[6] In [34], $Z_{\rm tensor}$ was reproduced from 5d SYM approach, in eq. (3.78) there, with extra two terms $\propto v + v^{-1}$ in the numerator. This part corresponds to a free 6d hypermultiplet which in fact decouples from the 6d SCFT, but is sometimes included into the studies for convenience to study M5-M9 system. This is similar to sometimes including the free $(2,0)$ tensor multiplet to the A_{N-1} $(2,0)$ theory, to describe N M5-branes. In this paper, the term proportional to $v + v^{-1}$ in (3.78) of [34] will be sent to $Z_{\rm other}$ of (4.6).

4.1 Instanton partition function

To take into account the effect of the Wilson line which breaks E_8 down to SO(16), we have to make a shift of the fugacities by (4.5). We decide to express w', y_8' in terms of w, y_8. After inserting $y_8' = y_8 q^{-2}$ (or $e^{2\pi i m_8} \to e^{2\pi i m_8 - 2\pi i \tau}$) to the elliptic genera Z_n of section 3, one finds

$$Z_n(q, m_l', \epsilon_{1,2}) = \left(\frac{y_8}{q}\right)^n \tilde{Z}_{\rm n}(q, m_l, \epsilon_{1,2}), \tag{4.12}$$

with

$$\begin{aligned}
\tilde{Z}_1 &= \frac{1}{2}\left(-Z_{1(1)} + Z_{1(2)} + Z_{1(3)} - Z_{1(4)}\right) \\
\tilde{Z}_2 &= \frac{1}{2}Z_{2(0)} + \frac{1}{4}\left(-Z_{2(1)} - Z_{2(2)} + Z_{2(3)} + Z_{2(4)} - Z_{2(5)} - Z_{2(6)}\right) \\
\tilde{Z}_3 &= \frac{1}{4}\left(-Z_{3(1)} - Z_{3(2)} + Z_{3(3)} + Z_{3(4)}\right) + \frac{1}{8}\left(-Z_{3(1)'} - Z_{3(2)'} + Z_{3(3)'} + Z_{3(4)'}\right) \\
\tilde{Z}_4 &= \frac{1}{8}\left(Z_{4(1)} - Z_{4(2)} - Z_{4(2)'} - Z_{4(3)} - Z_{4(3)'} + Z_{4(4)} + Z_{4(4)'}\right) + \frac{1}{16}Z_{4(1)'}\,,
\end{aligned} \tag{4.13}$$

and so on, where $Z_{n(i)}$'s are all defined and computed in section 3 as functions of $q, m_l, \epsilon_{1,2}$. In all $Z_{n(i)}$ on the right hand side, the arguments are y_8, not y_8'. The overall factors $\left(y_8 q^{-1}\right)^n$ in (4.12) cancel with the shift $w' = wqy_8^{-1}$ in $Z = \sum_{n=0}^{\infty}(w')^n Z_n$. Namely, the E_8 mass shift is inducing a different value of 2d theta angle, by changing various signs in (4.13). We compute $\tilde{f}(w, q, \epsilon_{1,2}, m_i)$ defined by

$$Z_{\rm QFT} \equiv Z_{\rm tensor} \sum_{n=0}^{\infty} w^n \tilde{Z}_n(q, \epsilon_{1,2}, m_i) = PE\left[\tilde{f}\right] \equiv \exp\left[\sum_{n=1}^{n} \frac{1}{n}\tilde{f}(w^n, q^n, n\epsilon_1, n\epsilon_2, nm_l)\right], \tag{4.14}$$

and expand $\tilde{f} = \sum_{n=0}^{\infty} w^n \tilde{f}_n(q, \epsilon_{1,2}, m_i)$. The results up to $\mathcal{O}(q^5)$ are as follows. $\tilde{f}_0$ at zero string number has been computed from the 5d calulus in [34], and agrees with g appearing in (4.10). So we consider $\tilde{f}_n$ with $n \geq 1$.

Defining $t \equiv e^{i\pi\epsilon_1 + i\pi\epsilon_2}$, $u \equiv e^{i\pi\epsilon_1 - i\pi\epsilon_2}$, $\tilde{f}_1$ is given by $\frac{t}{(1-tu)(1-t/u)}$ times

$$\begin{aligned}
&+q^0 \cdot \chi_{\mathbf{16}}^{\rm SO(16)} + q^1 \cdot \chi_{\mathbf{128}}^{\rm SO(16)} \\
&+q^2\left[(t+t^{-1})(u+u^{-1})\chi_{\mathbf{16}}^{\rm SO(16)} + \chi_{\mathbf{560}}^{\rm SO(16)} + \chi_{\mathbf{16}}^{\rm SO(16)}\right] + q^3\left[(t+t^{-1})(u+u^{-1})\chi_{\mathbf{128}}^{\rm SO(16)} + \chi_{\mathbf{1920}}^{\rm SO(16)} + \chi_{\mathbf{128}}^{\rm SO(16)}\right] \\
&+q^4\Big[(t+t^{-1})(u+u^{-1})(\chi_{\mathbf{560}}^{\rm SO(16)} + 2\chi_{\mathbf{16}}^{\rm SO(16)}) + \left((t^2+1+t^{-2})(u^2+1+u^{-2}) - 1\right)\chi_{\mathbf{16}}^{\rm SO(16)} \\
&\qquad + \chi_{\mathbf{4368}}^{\rm SO(16)} + \chi_{\mathbf{1344}}^{\rm SO(16)} + \chi_{\mathbf{560}}^{\rm SO(16)} + 4\chi_{\mathbf{16}}^{\rm SO(16)}\Big] \\
&+q^5\Big[(t+t^{-1})(u+u^{-1})(\chi_{\mathbf{1920}}^{\rm SO(16)} + 2\chi_{\mathbf{128}}^{\rm SO(16)}) + \left((t^2+1+t^{-2})(u^2+1+u^{-2}) - 1\right)\chi_{\mathbf{128}}^{\rm SO(16)} \\
&\qquad + \chi_{\mathbf{13312}}^{\rm SO(16)} + 2\chi_{\mathbf{1920}}^{\rm SO(16)} + 4\chi_{\mathbf{128}}^{\rm SO(16)}\Big] + \mathcal{O}(q^6)
\end{aligned} \tag{4.15}$$

The boldfaced subscripts are the irreps of SO(16) $\subset E_8$ in the 5d Sp(1) gauge theory with 8 fundamental flavors. $\chi_{\mathbf{R}}^{\rm SO(16)}$ is the SO(16) character of the representation $\mathbf{R}$. We computed $Z_{\rm QFT}$ of the 5d SYM, following the procedures outlined above (explained in [34]), up to five instantons. We further expanded it in the Coulomb VEV parameter to extract the $\mathcal{O}(w^1)$ order. This completely agrees with (4.15).

$\tilde{f}_2$ is given by $\frac{t}{(1-tu)(1-t/u)}$ times

$$
\begin{aligned}
&-q^0\cdot(t+t^{-1})-q^1\Big[(t+t^{-1})\chi^{\mathrm{SO}(16)}_{\mathbf{128}}\Big] \qquad (4.16)\\
&-q^2\Big[(t^3+t+t^{-1}+t^{-3})(u^2+1+u^{-2})+(u+u^{-1})+(t^2+1+t^{-2})(u+u^{-1})(\chi^{\mathrm{SO}(16)}_{\mathbf{120}}+1)\\
&\quad+(t+t^{-1})(\chi^{\mathrm{SO}(16)}_{\mathbf{1820}}+\chi^{\mathrm{SO}(16)}_{\mathbf{120}}+2)\Big]\\
&-q^3\Big[(t+t^{-1})((t^2+t^{-2})(u^2+1+u^{-2})-1)\chi^{\mathrm{SO}(16)}_{\mathbf{128}}+(u+u^{-1})\chi^{\mathrm{SO}(16)}_{\mathbf{128}}\\
&\quad+(t^2+1+t^{-2})(u+u^{-1})(\chi^{\mathrm{SO}(16)}_{\mathbf{1920}}+2\chi^{\mathrm{SO}(16)}_{\mathbf{128}})+(t+t^{-1})(\chi^{\mathrm{SO}(16)}_{\mathbf{13312}}+\chi^{\mathrm{SO}(16)}_{\mathbf{1920}}+4\chi^{\mathrm{SO}(16)}_{\mathbf{128}})\Big]\\
&-q^4\Big[(t^4+t^{-4})(u+u^{-1})+(t^3+t+t^{-1}+t^{-3})(u^4+u^{-4})\\
&\quad+(t^2+1+t^{-2})(u^3+u^{-3})+(t+t^{-1})(u^2+u^{-2})+(t^5+t^{-5})(u^4+u^2+1+u^{-2}+u^{-4})\\
&\quad+(u+u^{-1})(\chi^{\mathrm{SO}(16)}_{\mathbf{1820}}+2\chi^{\mathrm{SO}(16)}_{\mathbf{120}}+3)+\big((t^4+t^2+1+t^{-2}+t^{-4})(u^3+u^{-3})\\
&\quad+(t^4+t^{-4})(u+u^{-1})+(t^3+t^{-3})+(t+t^{-1})(u^2+u^{-2})\big)\,(\chi^{\mathrm{SO}(16)}_{\mathbf{120}}+1)\\
&\quad+\big((t^3+t^{-3})(u^2+1+u^{-2})+(t+t^{-1})(u^2+u^{-2})\big)\,(\chi^{\mathrm{SO}(16)}_{\mathbf{1820}}+\chi^{\mathrm{SO}(16)}_{\mathbf{135}}+2\chi^{\mathrm{SO}(16)}_{\mathbf{120}}+5)\\
&\quad+(t^2+1+t^{-2})(u+u^{-1})(\chi^{\mathrm{SO}(16)}_{\mathbf{8008}}+\chi^{\mathrm{SO}(16)}_{\mathbf{7020}}+2\chi^{\mathrm{SO}(16)}_{\mathbf{1820}}+\chi^{\mathrm{SO}(16)}_{\mathbf{135}}+6\chi^{\mathrm{SO}(16)}_{\mathbf{120}}+8)\\
&\quad+(t+t^{-1})(\chi^{\mathrm{SO}(16)}_{\mathbf{60060}}+\chi^{\mathrm{SO}(16)}_{\mathbf{8008}}+\chi^{\mathrm{SO}(16)}_{\mathbf{7020}}+\chi^{\mathrm{SO}(16)}_{\underline{\mathbf{6435}}}+\chi^{\mathrm{SO}(16)}_{\mathbf{5304}}+4\chi^{\mathrm{SO}(16)}_{\mathbf{1820}}+3\chi^{\mathrm{SO}(16)}_{\mathbf{135}}+9\chi^{\mathrm{SO}(16)}_{\mathbf{120}}+14)\Big]\\
&-q^5\Big[\,\big((t^5+t^{-5})(u^4+u^2+1+u^{-2}+u^{-4})+(t^3+t+t^{-1}+t^{-3})(u^4+u^{-4})\\
&\quad+(t^2+1+t^{-2})(u^3+u^{-3})+(t^4+t^{-4})(u+u^{-1})+(t+t^{-1})(u^2+u^{-2})\big)\,\chi^{\mathrm{SO}(16)}_{\mathbf{128}}\\
&\quad+\big((t^3+t^{-3})(u^2+1+u^{-2})+(t+t^{-1})(u^2+u^{-2})\big)\,(\chi^{\mathrm{SO}(16)}_{\mathbf{13321}}+3\chi^{\mathrm{SO}(16)}_{\mathbf{1920}}+7\chi^{\mathrm{SO}(16)}_{\mathbf{128}})\\
&\quad+\big((t^2+t^{-2})(u+u^{-1})+(t+t^{-1})+(u+u^{-1})\big)\,(\chi^{\mathrm{SO}(16)}_{\mathbf{56320}}+\chi^{\mathrm{SO}(16)}_{\mathbf{15360}}+3\chi^{\mathrm{SO}(16)}_{\mathbf{13312}}+7\chi^{\mathrm{SO}(16)}_{\mathbf{1920}}+14\chi^{\mathrm{SO}(16)}_{\mathbf{128}})\\
&\quad+(u+u^{-1})(\chi^{\mathrm{SO}(16)}_{\mathbf{13312}}+2\chi^{\mathrm{SO}(16)}_{\mathbf{1920}}+5\chi^{\mathrm{SO}(16)}_{\mathbf{128}})+\big((t^2+1+t^{-2})(u^3+u^{-3})\\
&\quad+(t^4+t^{-4})(u^3+u+u^{-1}+u^{-3})+(t+t^{-1})(u^2+u^{-2})+(t^3+t^{-3})\big)\,(\chi^{\mathrm{SO}(16)}_{\mathbf{1920}}+2\chi^{\mathrm{SO}(16)}_{\mathbf{128}})\\
&\quad+(t+t^{-1})(\chi^{\mathrm{SO}(16)}_{\mathbf{161280}}+\chi^{\mathrm{SO}(16)}_{\mathbf{141440}}+3\chi^{\mathrm{SO}(16)}_{\mathbf{13312}}+5\chi^{\mathrm{SO}(16)}_{\mathbf{1920}}+9\chi^{\mathrm{SO}(16)}_{\mathbf{128}})\Big]+\mathcal{O}(q^6)
\end{aligned}
$$

This again agrees with the result obtained from the instanton calculus of [34].

We also computed $\tilde{f}_3$ with all $\mathrm{SO}(16)\subset E_8$ masses turned off. It again completely agrees with $\tilde{f}_3$ computed from 5d instanton calculus, up to q^5 order that we checked. Also, for 3 and 4 E-strings, we have kept all E_8 masses and compared our 2d elliptic genus with the instanton partition function up to 1 instanton order, which all show agreements.

So we saw that the instanton calculus provides the correct index for the E_8 6d SCFT. One virtue of this approach would be that, at a given order in q, the index is computed exactly in w. In particular, the chemical potential for the E-string number (the Coulomb VEV of 5d SYM) is an integration variable in the curved space partition functions, which can be used to study the conformal field theory physics. So knowing the exact form of the partition function in w will be desirable to understand the curved space partition functions.

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5 Concluding remarks

In this paper we have found a description of E-strings which can be used to describe the IR degrees of freedom on it. This in particular includes the information about bound states of E-strings. The theory for n E-strings involves a $(0,4)$ supersymmetric quiver theory in 2 dimensions with $O(n)$ gauge symmetry and some matter content. We in particular computed the elliptic genus of E-strings (including turning on fugacities for the E_8 flavor symmetry as well as SO(4) rotation transverse to the string in 6d) for small number of E-strings. We gave the explicit answer for $n = 1, 2, 3, 4$ and indicated how one can use these methods to obtain arbitrary n answers. Our results successfully pass the comparison checks with the partial results already known. Our results provide an all genus answer for the topological string on the canonical bundle over $\frac{1}{2}K3$. In addition, we explained how to compute the same elliptic genus using the instanton partition function of the 5d Sp(1) SYM theory coupled to 8 fundamental hypermultiplets.

We briefly discuss various physics of E-strings that we can learn from our gauge theories and the elliptic genus formula. Firstly, one can show from our contour integral expression (3.11), (3.10) and $\frac{\eta(-1/\tau)}{\theta_1(-1/\tau,z/\tau)} = \varepsilon e^{-\frac{\pi i z^2}{\tau}} \frac{\eta(\tau)}{\theta_1(\tau,z)}$ (where ε is a z independent phase) that

$$Z_n\left(-\frac{1}{\tau}, \frac{\epsilon_{1,2}}{\tau}, \frac{m_l}{\tau}\right) = Z_n(\tau, \epsilon_{1,2}, m_l)\cdot\varepsilon^{-6n} \exp\left[\frac{\pi i}{\tau}\left(2\epsilon_1\epsilon_2 n^2 - \left(\sum_{l=1}^{8} m_l^2 - 4\epsilon_+^2\right) n\right)\right]. \quad (5.1)$$

This expression can be obtained by applying the S-modular transformation directly to the integrand (3.10), noting that the transformation just shuffles the discrete holonomy sectors with the same dimension for their Weyl groups. In fact, the extra exponential factor on the right hand side is related to the 2d 't Hooft anomaly on the strings [24], being $\exp\left[-\frac{\pi i}{\tau}\mathcal{A}^{ab}u_a u_b\right]$ with chemical potentials u_a when the 't Hooft anomaly is given by $\mathcal{A}^{ab} = \sum_{\text{fermions}} \gamma_3 K^a K^b$. Thus, there are terms in the anomalies which are linear in the string number n, and also a peculiar term which is proportional to n^2.

The last term proportional to n^2 has interesting physical implications to the non-linear sigma models in IR that one obtains from our gauge theories. Namely, the real $4n$ dimensional solution for $\varphi, \tilde{\varphi}$ which solves $\varphi\tilde{\varphi} - \tilde{\varphi}\varphi = 0$, $\varphi\varphi^\dagger - \tilde{\varphi}^\dagger\tilde{\varphi} = 0$ of section 2 is given by diagonal matrices for $\varphi, \tilde{\varphi}$. By extra modding out by the unbroken gauge symmetries in $O(n)$, the moduli space becomes the n'th symmetric product of $\mathbb{R}^4$, $\text{Sym}^n(\mathbb{R}^4) = (\mathbb{R}^4)^n/S_n$ where S_n is the n dimensional permutation group. Considering the non-linear sigma model on this target space, away from the singularity, there are no ways to have anomalies (or any other measures of degree of freedom) which scale like n^2, since the number of degrees of freedom visible in the sigma model is proportional to n. Therefore, the extra n^2 degrees of freedom which contribute to the first term in the anomaly should be supported at the orbifold singularity, where the sigma model description should break down.

This is in contrast to the dynamics of fundamental strings. Namely, if one wraps the fundamental string on a circle n times, its dynamics on the transverse target space is also described by n'th symmetry product of the transverse space. So although the non-linear sigma models for our E-strings apparently looks similar to those for the fundamental strings, the way one treats the orbifold singularity should be very different. For instance, for a fundamental superstring, the elliptic genus Z_n for n wrapped strings is given in terms of the elliptic genus Z_1 of the single string, by the Hecke transformation. Namely, defining the grand partition function

$$Z(w,\tau,z)=\sum_{n=0}^{\infty}Z_n(\tau,z)w^n \tag{5.2}$$

where z collectively denotes chemical potentials, and $Z_0\equiv 1$ by definition, $Z(w,\tau,z)$ is given in terms of Z_1 by [54]

$$Z(w,\tau,z)=\exp\left[\sum_{n=1}^{\infty}\frac{1}{n}w^n\sum_{ad=n;a,d\in\mathbb{Z}}\sum_{b(\text{mod }d)}Z_1\left(\frac{a\tau+b}{d},az\right)\right]\equiv\exp\left[\sum_{n=1}^{\infty}w^nT_nZ_1(\tau,z)\right], \tag{5.3}$$

where T_n are the Hecke operators. This expresses all Z_n's in terms of Z_1. For instance, Z_2 for fundamental strings is given from this relation by

$$Z_2(\tau,z)=\frac{1}{2}\left[Z_1(\tau,z)^2+Z(2\tau,2z)+Z_1\left(\frac{\tau}{2},z\right)+Z_1\left(\frac{\tau+1}{2},z\right)\right]. \tag{5.4}$$

Now with the extra anomalies on the E-strings proportional to n^2, it is easy to understand that the elliptic genera Z_n at $n>1$ cannot be expressed in terms of Z_1 by Hecke transformation. This is because, from the formula (5.3), the 2d anomaly has to be additive. The additive property means that, if Z_1 has the anomaly $\exp\left[-\frac{\pi i}{\tau}\mathcal{A}^{ab}u_au_b\right]$ under S-modular transformation like (5.1), Z_n given by (5.3) should have anomaly $\exp\left[-\frac{n\pi i}{\tau}\mathcal{A}^{ab}u_au_b\right]$. However, since (5.1) for E-strings exhibits an anomaly proportional to n^2, (5.3) cannot be true for E-strings.

It is easy to see, from the viewpoint of our 2d gauge theory, how the non-linear sigma model description breaks down near the singularity, and how the n^2 degrees of freedom emerges at the singularity. When $\varphi,\tilde{\varphi}$ assume large nonzero values, the fermions in the $O(n)$ vector multiplet (which we called $\lambda_+^{\dot{\alpha}A}$, with $\frac{n^2-n}{2}$ components) become massive, so do not appear in the non-linear sigma model. However, since gauge symmetry is unbroken at $\varphi=\tilde{\varphi}=0$, these fermions become light near the orbifold singularity. The left-moving fermion λ_+ acquires mass only by combining with right-moving fermions, which are $\lambda_-^{\alpha A}$ of section 2 (superpartners of $\varphi,\tilde{\varphi}$). Both λ_+,λ_- become light near the singularity, and the anomaly in (5.1) proportional to $\epsilon_1\epsilon_2n^2$ precisely comes from these fields in our UV description. Namely, a crucial difference between fundamental strings and our E-strings (and more generally other self-dual strings of 6d SCFTs in the tensor branch) can be explained with gauge theory engineering of the latter.

As mentioned in section 3.1, another curious aspect of E-strings can be explained using our gauge theory descriptions. The elliptic genus of single strings have been computed in [19] using an effective free string theory approach, where the GSO projection (like that of the $E_8\times E_8$ heterotic strings) had to be applied on R-NS sectors to get the correct results. From our $O(1)\sim\mathbb{Z}_2$ gauge theory approach, these are simply the consequence of summing over all the discrete $O(1)$ flat connections on T^2. This observation generalizes to all $O(n)$ elliptic genera, as we have elaborated in the earlier part of section 3, by having 7 discrete sectors for $n = 2$ and 8 sectors for $n \geq 3$. It is possible that understanding these structures more directly could be a key question to better understand the IR conformal field theories on these strings.

With these interesting physics in mind, let us close this paper by addressing a few natural extensions of the present work. First of all it would be nice to see if we can streamline the computation of the elliptic genus for arbitrary n. Even though our methods provide an answer, writing it explicitly is cumbersome. Secondly, it would be interesting to see if we can find an explicit description of the $(0, 4)$ conformal theory they flow to. Finally it would be interesting to see if we can use our results to come up with a domain wall description of the E-string amplitude as in [7]. Moreover one would like to use this to show that the partition function of a pair of n E-strings can lead to the partition function of n heterotic strings as is predicted by the Horava-Witten description of heterotic string. Finally it would be interesting to generalize this to other $(1, 0)$ superconformal field theories in 6d, and characterize all the 2d $(0, 4)$ systems that one gets on the worldsheet of the associated strings.

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A Modular forms and Jacobi forms

A modular form $f_n(\tau)$ of weight n transforms under $\mathrm{SL}(2, \mathbb{Z})$ as

$$f_n\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^n f_n(\tau) \ , \quad ad - bc = 1\,. \tag{A.1}$$

An important class of modular forms is given by the Eisenstein series,

$$E_{2k}(\tau) = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n, \tag{A.2}$$

where $q = e^{2\pi i\tau}$. The Bernoulli numbers B_{2k} and the divisor functions $\sigma_k(n)$ are defined by

$$\sum_{k=0}^{\infty} B_k \frac{x^k}{k!} = \frac{x}{e^x - 1}\,, \quad \sigma_k(n) = \sum_{d|n} d^k. \tag{A.3}$$

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$E_{2k}(\tau)$ are modular forms of weight $2k$, expect for $E_2(\tau)$ which involves an anomalous term,

$$E_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 E_2(\tau) + \frac{6}{i\pi}c(c\tau+d). \tag{A.4}$$

Another example of modular form is the Dedekind eta function $\eta(\tau)$, defined by

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)\,. \tag{A.5}$$

Under the modular transformation, $\eta(\tau)$ behaves as a weight $\frac{1}{2}$ form up to a phase $\epsilon(a,b,c,d)$,

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = \epsilon(a,b,c,d)\cdot(c\tau+d)^{1/2}\eta(\tau). \tag{A.6}$$

Jacobi forms have a modular parameter τ and an elliptic parameter z. Modular transformation for Jacobi forms $\phi_{k,m}(\tau,z)$ of weight k and index m is given by

$$\phi_{k,m}\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right) = (c\tau+d)^k e^{\frac{2\pi imcz^2}{c\tau+d}}\phi_{k,m}(\tau,z), \tag{A.7}$$

Under the translation of the elliptic parameter z, they behave as

$$\phi_{k,m}(\tau,z+a\tau+b) = e^{-2\pi im(a^2\tau+2az)}\phi_{k,m}(\tau,z). \tag{A.8}$$

where a,b are integers.

The Jacobi theta function $\vartheta(\tau,z)$ is a Jacobi form of weight $\frac{1}{2}$ and index $\frac{1}{2}$, defined as

$$\vartheta(\tau,z) = \prod_{n=1}^{\infty}(1-q^n)(1+q^{n-\frac{1}{2}}y)(1+q^{n-\frac{1}{2}}y^{-1}) = \sum_{n\in\mathbb{Z}}q^{n^2/2}y^n \tag{A.9}$$

where $q\equiv e^{2\pi i\tau}$ and $y\equiv e^{2\pi iz}$. We define three other functions which are closely related to the Jacobi theta function, and define

$$\begin{aligned}
\theta_1(\tau,z) &= -iq^{1/8}y^{1/2}\vartheta(\tau,z+\tfrac{1+\tau}{2}) = -iq^{1/8}y^{1/2}\prod_{n=1}^{\infty}(1-q^n)(1-q^n y)(1-q^{n-1}y^{-1})\\
\theta_2(\tau,z) &= q^{1/8}y^{1/2}\vartheta(\tau,z+\tfrac{\tau}{2}) = q^{1/8}y^{1/2}\prod_{n=1}^{\infty}(1-q^n)(1+q^n y)(1+q^{n-1}y^{-1})\\
\theta_3(\tau,z) &= \vartheta(\tau,z) = \prod_{n=1}^{\infty}(1-q^n)(1+q^{n-\frac{1}{2}}y)(1+q^{n-\frac{1}{2}}y^{-1})\\
\theta_4(\tau,z) &= \vartheta(\tau,z+\tfrac{1}{2}) = \prod_{n=1}^{\infty}(1-q^n)(1-q^{n-\frac{1}{2}}y)(1-q^{n-\frac{1}{2}}y^{-1}).
\end{aligned} \tag{A.10}$$

From here, when we omit the modular parameter in various functions, it should be understood as τ. $\theta_n(z)$'s are related to others by the half-period shifts:

$$\begin{array}{lll}
\theta_1(z+\frac{1}{2}) = \theta_2(z) & \theta_1(z+\frac{1+\tau}{2}) = q^{-1/8}y^{-1/2}\theta_3(z) & \theta_1(z+\frac{\tau}{2}) = iq^{-1/8}y^{-1/2}\theta_4(z)\\
\theta_2(z+\frac{1}{2}) = -\theta_1(z) & \theta_2(z+\frac{1+\tau}{2}) = -iq^{-1/8}y^{-1/2}\theta_4(z) & \theta_2(z+\frac{\tau}{2}) = q^{-1/8}y^{-1/2}\theta_3(z)\\
\theta_3(z+\frac{1}{2}) = \theta_4(z) & \theta_3(z+\frac{1+\tau}{2}) = iq^{-1/8}y^{-1/2}\theta_1(z) & \theta_3(z+\frac{\tau}{2}) = q^{-1/8}y^{-1/2}\theta_2(z)\\
\theta_4(z+\frac{1}{2}) = \theta_3(z) & \theta_4(z+\frac{1+\tau}{2}) = q^{-1/8}y^{-1/2}\theta_2(z) & \theta_4(z+\frac{\tau}{2}) = iq^{-1/8}y^{-1/2}\theta_1(z)
\end{array} \tag{A.11}$$

Various identities. The modular forms E_4, E_6, and η can be expressed in terms of Jacobi theta functions with their elliptic parameters z set to zero:

$$\begin{aligned} E_4 &= \frac{1}{2}(\theta_2(0)^8 + \theta_3(0)^8 + \theta_4(0)^8) \\ E_6 &= \frac{1}{2}(\theta_2(0)^4 + \theta_3(0)^4)(\theta_3(0)^4 + \theta_4(0)^4)(\theta_4(0)^4 - \theta_2(0)^4) \\ 2\eta^3 &= \theta_2(0)\theta_3(0)\theta_4(0). \end{aligned} \tag{A.12}$$

$\theta_n(z)$'s also satisfy

$$\theta_2(z)^4 - \theta_1(z)^4 = \theta_3(z)^4 - \theta_4(z)^4\,, \quad \theta_2(0)^4 = \theta_3(0)^4 - \theta_4(0)^4\,. \tag{A.13}$$

Further identities of $\theta_n(z)$'s with different elliptic parameters are

$$\begin{aligned} &\theta_1(a+b)\theta_1(a-b)\theta_4(0)^2 = \theta_3(a)^2\theta_2(b)^2 - \theta_2(a)^2\theta_3(b)^2 = \theta_1(a)^2\theta_4(b)^2 - \theta_4(a)^2\theta_1(b)^2 \\ &\theta_3(a+b)\theta_3(a-b)\theta_2(0)^2 = \theta_3(a)^2\theta_2(b)^2 + \theta_4(a)^2\theta_1(b)^2 = \theta_2(a)^2\theta_3(b)^2 + \theta_1(a)^2\theta_4(b)^2 \\ &\theta_3(a+b)\theta_3(a-b)\theta_3(0)^2 = \theta_1(a)^2\theta_1(b)^2 + \theta_3(a)^2\theta_3(b)^2 = \theta_2(a)^2\theta_2(b)^2 + \theta_4(a)^2\theta_4(b)^2 \\ &\theta_3(a+b)\theta_3(a-b)\theta_4(0)^2 = \theta_4(a)^2\theta_3(b)^2 - \theta_1(a)^2\theta_2(b)^2 = \theta_3(a)^2\theta_4(b)^2 - \theta_2(a)^2\theta_1(b)^2 \end{aligned} \tag{A.14}$$

$$\begin{aligned} &\theta_1(a \pm b)\theta_2(a \mp b)\theta_3(0)\theta_4(0) = \theta_1(a)\theta_2(a)\theta_3(b)\theta_4(b) \pm \theta_3(a)\theta_4(a)\theta_1(b)\theta_2(b) \\ &\theta_1(a \pm b)\theta_3(a \mp b)\theta_2(0)\theta_4(0) = \theta_1(a)\theta_3(a)\theta_2(b)\theta_4(b) \pm \theta_2(a)\theta_4(a)\theta_1(b)\theta_3(b) \\ &\theta_1(a \pm b)\theta_4(a \mp b)\theta_2(0)\theta_3(0) = \theta_1(a)\theta_4(a)\theta_2(b)\theta_3(b) \pm \theta_2(a)\theta_3(a)\theta_1(b)\theta_4(b)\,. \end{aligned} \tag{A.15}$$

Remaining identities of this kind can be obtained through half-period shifts on a.

Under the shift of modular parameter $\tau \to \tau' = \tau + 1$, the corresponding changes are

$$\theta_1(\tau+1,z) = e^{i\frac{\pi}{4}}\theta_1(\tau,z),\ \theta_2(\tau+1,z) = e^{i\frac{\pi}{4}}\theta_2(\tau,z),\ \theta_3(\tau+1,z) = \theta_4(\tau,z),\ \theta_4(\tau+1,z) = \theta_3(\tau,z). \tag{A.16}$$

Watson's identities and Landen's formulas involve doubling of modular parameter τ,

$$\begin{aligned} \theta_1(\tau,z)\theta_1(\tau,w) &= \theta_3(2\tau,z+w)\theta_2(2\tau,z-w) - \theta_2(2\tau,z+w)\theta_3(2\tau,z-w) \\ \theta_3(\tau,z)\theta_3(\tau,w) &= \theta_3(2\tau,z+w)\theta_3(2\tau,z-w) + \theta_2(2\tau,z+w)\theta_2(2\tau,z-w) \end{aligned} \tag{A.17}$$

$$\begin{aligned} \theta_1(2\tau,2z) &= \theta_1(\tau,z)\theta_2(\tau,z)/\theta_4(2\tau,0) \\ \theta_4(2\tau,2z) &= \theta_3(\tau,z)\theta_4(\tau,z)/\theta_4(2\tau,0)\,. \end{aligned} \tag{A.18}$$

Considering these identities at $z = 0$ or $z = w = 0$, and also using the second identity of (A.13), one obtains

$$\theta_2(2\tau,0) = \sqrt{\tfrac{\theta_3(\tau,0)^2 - \theta_4(\tau,0)^2}{2}},\ \theta_3(2\tau,0) = \sqrt{\tfrac{\theta_3(\tau,0)^2 + \theta_4(\tau,0)^2}{2}},\ \theta_4(2\tau,0) = \sqrt{\theta_3(\tau,0)\theta_4(\tau,0)}. \tag{A.19}$$

Differentiations by τ, z. The τ derivatives of E_2, E_4, E_6 can be obtained from the Ramanujan identities

$$q\frac{d}{dq}E_2 = \frac{1}{12}(E_2^2 - E_4),\ q\frac{d}{dq}E_4 = \frac{1}{3}(E_2E_4 - E_6),\ q\frac{d}{dq}E_6 = \frac{1}{2}(E_2E_6 - E_4^2). \tag{A.20}$$

The τ derivative of the eta function is given by

$$q\frac{d}{dq}\eta^3 = \frac{\eta^3}{8}E_2. \tag{A.21}$$

As for the theta functions, first note that $\theta_n(z)$'s are solutions of

$$\left[\frac{1}{(2\pi i)^2}\frac{\partial^2}{\partial z^2} - \frac{1}{i\pi}\frac{\partial}{\partial\tau}\right]\theta_n(\tau,z) = \left[\frac{1}{(2\pi i)^2}\frac{\partial^2}{\partial z^2} - 2q\frac{\partial}{\partial q}\right]\theta_n(\tau,z) = 0. \tag{A.22}$$

θ_1 is an odd function of z, while $\theta_2, \theta_3, \theta_4$ are even functions of z. The lowest non-vanishing derivatives of θ_n's at $z = 0$ are given by

$$\begin{aligned}
&\theta_1^{(1)}(0) = 2\pi\eta^3 && \theta_2^{(2)}(0) = -\frac{\pi^2}{3}\theta_2(0)\left(E_2 + \theta_3(0)^4 + \theta_4(0)^4\right)\\
&\theta_3^{(2)}(0) = -\frac{\pi^2}{3}\theta_3(0)\left(E_2 + \theta_2(0)^4 - \theta_4(0)^4\right) && \theta_4^{(2)}(0) = -\frac{\pi^2}{3}\theta_4(0)\left(E_2 - \theta_2(0)^4 - \theta_3(0)^4\right),
\end{aligned} \tag{A.23}$$

where the superscript (n) denotes n'th derivative with respect to the elliptic parameter. Using (A.22), (A.23), (A.20) and (A.21), one can also express the higher z derivatives $\theta_1^{(2n+1)}(0)$, $\theta_2^{(2n)}(0)$, $\theta_3^{(2n)}(0)$, $\theta_4^{(2n)}(0)$ at $z = 0$ in terms of $\theta_2(0)$, $\theta_3(0)$, $\theta_4(0)$, E_2. See appendix C for more details, where this procedure will be illustrated and used to prove exact properties of the E-string elliptic genera.

B Genus expansions of topological string amplitudes

In this appendix, we summarize some low genus results that we used in section 3. The low genus amplitudes have been studied in [19, 29, 30, 43, 55]. We list the unrefined results till $g \leq 5$ (as written in [43]), and some refined results that we used to compare with our results.

For three E-strings, the unrefined genus expansion coefficients $F^{(0,g,3)}$ are given by

$$\begin{aligned}
F^{(0,0,3)} &= \frac{54E_2^2E_4^3 + 216E_2E_4^2E_6 + 109E_4^4 + 197E_4E_6^2}{15552\eta^{36}}\\
F^{(0,1,3)} &= \frac{78E_2^3E_4^3 + 299E_2E_4^4 + 360E_2^2E_4^2E_6 + 472E_4^3E_6 + 439E_2E_4E_6^2 + 80E_6^3}{62208\eta^{36}}\\
F^{(0,2,3)} &= \frac{1}{2488320\eta^{36}}\left(575E_2^4E_4^3 + 3040E_2^3E_4^2E_6 + 4690E_2^2E_4E_6^2 + 3548E_2^2E_4^4\right.\\
&\qquad\left. + 1600E_6^3E_2 + 10176E_6E_4^3E_2 + 2231E_4^5 + 5244E_4^2E_6^2\right)\\
F^{(0,3,3)} &= \frac{1}{209018880\eta^{36}}\left(138104E_4^4E_6 + 224024E_6E_4^3E_2^2 + 36400E_2^4E_4^2E_6 + 224456E_4^2E_6^2E_2\right.\\
&\quad\left. + 49584E_4E_6^3 + 68460E_2^3E_4E_6^2 + 55006E_2^3E_4^4 + 6055E_2^5E_4^3 + 97431E_4^5E_2 + 33600E_6^3E_2^2\right)\\
F^{(0,4,3)} &= \frac{1}{75246796800\eta^{36}}\left(3164700E_2^4E_4E_6^2 + 8993259E_4^5E_2^2 + 14111840E_6^2E_4^3 + 806400E_6^4\right.\\
&\quad + 25171632E_2E_6E_4^4 + 13855280E_2^3E_6E_4^3 + 8963520E_2E_6^3E_4 + 20453520E_2^2E_6^2E_4^2\\
&\quad\left. + 4014627E_4^6 + 208985E_2^6E_4^3 + 2016000E_6^3E_2^3 + 1417920E_2^5E_4^2E_6 + 2638125E_2^4E_4^4\right)\\
F^{(0,5,3)} &= \frac{1}{9932577177600\eta^{36}}\left(935093824E_6^2E_4^3E_2 + 233170300E_2^4E_6E_4^3 + 296640960E_2^2E_6^3E_4\right.\\
&\quad + 837550728E_2^2E_6E_4^4 + 453680480E_2^3E_6^2E_4^2 + 16385600E_2^6E_4^2E_6 + 42513240E_2^5E_4E_6^2\\
&\quad + 201151929E_4^5E_2^3 + 36275085E_2^5E_4^4 + 53222400E_6^4E_2 + 266767491E_4^6E_2\\
&\quad\left. + 405268284E_4^5E_6 + 268326944E_4^2E_6^3 + 33264000E_6^3E_2^4 + 2155615E_2^7E_4^3\right).
\end{aligned} \tag{B.1}$$

A refined coefficient $F^{(1,0,3)}$ that we studied in section 3.3 is given by

$$F^{(1,0,3)} = -\frac{54E_2^3E_4^3+235E_2E_4^4+216E_2^2E_4^2E_6+776E_4^3E_6+287E_2E_4E_6^2+160E_6^3}{124416\eta^{36}}\,. \qquad \text{(B.2)}$$

For the four E-strings, $F^{(0,g,4)}$ are given as follows (after correcting some typos in [43]):

$$\begin{aligned}
F^{(0,0,4)} &= \frac{1}{62208\eta^{48}}E_4\left(272E_4^3E_6+154E_6^3+109E_2E_4^4+269E_2E_4E_6^2+144E_2^2E_4^2E_6+24E_2^3E_4^3\right)\\
F^{(0,1,4)} &= \frac{1}{11943936\eta^{48}}\left(37448E_2^2E_4^2E_6^2+68768E_2E_4^4E_6+29920E_2E_4E_6^3+13809E_4^6\right.\\
&\quad\left.+57750E_4^3E_6^2+17416E_2^2E_4^5+4545E_6^4+16704E_2^3E_4^3E_6+2472E_2^4E_4^4\right)\\
F^{(0,2,4)} &= \frac{1}{179159040\eta^{48}}\left(77280E_2^4E_6E_4^3+209200E_2^2E_6^3E_4+547760E_2^2E_6E_4^4+214811E_4^6E_2\right.\\
&\quad+203900E_2^3E_6^2E_4^2+103252E_4^5E_2^3+827230E_6^2E_4^3E_2+10200E_2^5E_4^4+57375E_6^4E_2\\
&\quad\left.+420616E_4^5E_6+314360E_4^2E_6^3\right)\\
F^{(0,3,4)} &= \frac{1}{90296156160\eta^{48}}\left(28134630E_4^7+151049093E_4^4E_6^2+25488295E_4E_6^4+966630E_2^6E_4^4\right.\\
&\quad+189296376E_6^2E_4^3E_2^2+8172360E_2^5E_6E_4^3+31388000E_2^3E_6^3E_4+88718416E_2^3E_6E_4^4\\
&\quad+24977155E_2^4E_6^2E_4^2+13366787E_4^5E_2^4+12119625E_6^4E_2^2+137926976E_4^2E_6^3E_2\\
&\quad\left.+51557313E_4^6E_2^2+192353224E_4^5E_6E_2\right)\\
F^{(0,4,4)} &= \frac{1}{5417769369600\eta^{48}}\left(3336940980E_2^3E_4^3E_6^2+7817234620E_2E_6^2E_4^4+3248768730E_6^3E_4^3\right.\\
&\quad+5085796952E_2^2E_4^5E_6+101280375E_6^5+3550525000E_2^2E_4^2E_6^3+1290318725E_2E_4E_6^4\\
&\quad+936363912E_4^6E_2^3+1481276055E_4^7E_2+2912603799E_4^6E_6+1216807640E_2^4E_4^4E_6\\
&\quad+152620090E_2^5E_4^5+78676080E_2^6E_6E_4^3+410158000E_2^4E_6^3E_4+274844990E_2^5E_6^2E_4^2\\
&\quad\left.+8381520E_2^7E_4^4+202702500E_6^4E_2^3\right)\\
F^{(0,5,4)} &= \frac{1}{2860582227148800\eta^{48}}\left(12207942670E_2^6E_4^5+523849095E_2^8E_4^4+156150752805E_4^8\right.\\
&\quad+113811930320E_2^5E_4^4E_6+1311485716360E_4^6E_6E_2+1760563778482E_2^2E_6^2E_4^4\\
&\quad+286289201000E_2^2E_4E_6^4+381058740370E_2^4E_4^3E_6^2+1449394307792E_6^3E_4^3E_2\\
&\quad+1106487740990E_6^2E_4^5+44575839000E_6^5E_2+109025587484E_4^6E_2^4\\
&\quad+774483173328E_2^3E_4^5E_6+531170439360E_2^3E_4^2E_6^3+5431290480E_2^7E_6E_4^3\\
&\quad+37160939200E_2^5E_6^3E_4+337421738130E_4^7E_2^2+21439577390E_2^6E_6^2E_4^2\\
&\quad\left.+22344052500E_6^4E_2^4+344998537324E_6^4E_4^2\right)\,. \qquad \text{(B.3)}
\end{aligned}$$

C Exact properties of the E-string elliptic genus

We explain the details on how we checked various exact properties of our E-string elliptic genera, using the identities of appendix A. We made lots of symbolic computations using computer. Below, we explain how one can simplify various expressions which can be put on a computer for further simplifications.

2 E-strings. We compare the two expressions for the elliptic genus of 2 E-strings, (3.25) and (3.26). Let us denote them by Z_2 and $Z_2^{\rm E8}$ respectively, in the sense that the latter expression shows manifest E_8 symmetry. After setting $\epsilon_1 = -\epsilon_2 = \epsilon$ for simplicity, Z_2 is given by

$$\begin{aligned} Z_2 = \sum_{n=1}^{4} \frac{\prod_{l=1}^{8}\theta_n(m_l \pm \frac{\epsilon}{2})}{2\eta^{12}\theta_1(\epsilon)^2\theta_1(2\epsilon)^2} + \frac{1}{4\eta^{12}\theta_1(\epsilon)^4}\Bigg[\frac{\theta_2(0)^2}{\theta_2(\epsilon)^2}\left(\prod_{l=1}^{8}\theta_1(m_l)\theta_2(m_l) + \prod_{l=1}^{8}\theta_3(m_l)\theta_4(m_l)\right) \\ + \frac{\theta_4(0)^2}{\theta_4(\epsilon)^2}\left(\prod_{l=1}^{8}\theta_1(m_l)\theta_4(m_l) + \prod_{l=1}^{8}\theta_2(m_l)\theta_3(m_l)\right) \\ + \frac{\theta_3(0)^2}{\theta_3(\epsilon)^2}\left(\prod_{l=1}^{8}\theta_1(m_l)\theta_3(m_l) + \prod_{l=1}^{8}\theta_2(m_l)\theta_4(m_l)\right)\Bigg]. \end{aligned} \tag{C.1}$$

Using the identity (A.15) with $a = b$, one can write $Z_2 = \frac{N(\tau,z,m_l)}{\eta^{12}\theta_1(\epsilon)^2\theta_1(2\epsilon)^2}$ with

$$\begin{aligned} N = \sum_{n=1}^{4}\frac{1}{2}\prod_{l=1}^{8}\theta_n(m_l \pm \frac{\epsilon}{2}) + \frac{\theta_3(\epsilon)^2\theta_4(\epsilon)^2}{\theta_3(0)^2\theta_4(0)^2}\left(\prod_{l=1}^{8}\theta_1(m_l)\theta_2(m_l) + \prod_{l=1}^{8}\theta_3(m_l)\theta_4(m_l)\right) \\ + \frac{\theta_2(\epsilon)^2\theta_3(\epsilon)^2}{\theta_2(0)^2\theta_3(0)^2}\left(\prod_{l=1}^{8}\theta_1(m_l)\theta_4(m_l) + \prod_{l=1}^{8}\theta_2(m_l)\theta_3(m_l)\right) \\ + \frac{\theta_2(\epsilon)^2\theta_4(\epsilon)^2}{\theta_2(0)^2\theta_4(0)^2}\left(\prod_{l=1}^{8}\theta_1(m_l)\theta_3(m_l) + \prod_{l=1}^{8}\theta_2(m_l)\theta_4(m_l)\right). \end{aligned} \tag{C.2}$$

We apply (A.14) to the first term of N, where we take $a = m_l$, $b = \epsilon/2$. Then N can be expressed as a polynomial of $\theta_n(m_l)$, $\theta_n(\epsilon)$ and $\theta_n(\epsilon/2)$, with coefficients given by $\theta_n(0)$.

On the other hand, expressing (3.26) as $Z_2^{\rm E8} = N^{\rm E8}/(\eta^{12}\theta_1(\epsilon)^2\theta_1(2\epsilon)^2)$, we consider

$$\begin{aligned} N^{\rm E8} = \frac{1}{72}A_1^2(\phi_{0,1}(\epsilon)^2 - E_4\phi_{-2,1}(\epsilon)^2) + \frac{1}{96}A_2(E_4^2\phi_{-2,1}(\epsilon)^2 - E_6\phi_{-2,1}(\epsilon)\phi_{0,1}(\epsilon)) \\ + \frac{5}{288}B_2(E_6\phi_{-2,1}(\epsilon)^2 - E_4\phi_{-2,1}(\epsilon)\phi_{0,1}(\epsilon)). \end{aligned} \tag{C.3}$$

We first insert (A.12) to replace E_4, E_6, η by expressions containing $\theta_2(0)$, $\theta_3(0)$, $\theta_4(0)$ only. Looking at the definition of A_2 and B_2 in (3.28), there appear $\theta_n(\frac{\tau}{2}, m_l)$ and $\theta_n(\frac{\tau+1}{2}, m_l)$. To simplify them, we first consider the identities,

$$\theta_1(\tfrac{\tau}{2}, m_1)\theta_1(\tfrac{\tau}{2}, m_2) = \theta_3(\tau, m_1+m_2)\theta_2(\tau, m_1-m_2) - \theta_2(\tau, m_1+m_2)\theta_3(\tau, m_1-m_2) \tag{C.4}$$
$$\theta_1(\tfrac{\tau+1}{2}, m_1)\theta_1(\tfrac{\tau+1}{2}, m_2) = e^{i\pi/4}\theta_4(\tau, m_1+m_2)\theta_2(\tau, m_1-m_2) - e^{i\pi/4}\theta_2(\tau, m_1+m_2)\theta_4(\tau, m_1-m_2)\,.$$

The first identity can be obtained by replacing τ, z, w in (A.17) by $\frac{\tau}{2}, m_1, m_2$, respectively, and the second one is obtained from the first identity by using (A.16). One can also obtain three more copies of similar identities, replacing θ_1 on the left hand sides by $\theta_2, \theta_3, \theta_4$, by using (A.11). The expressions appearing on the right hand sides of (C.4) can be written as polynomials of $\theta_n(\tau, m_l)$ by using (A.15). We apply these identities, and also those with (m_1, m_2) replaced by (m_3, m_4), (m_5, m_6), (m_7, m_8), to (C.3). Then one can express all theta functions with modular parameters $\frac{\tau}{2}$ or $\frac{\tau+1}{2}$ in terms of $\theta_n(\tau, m_l)$. Other terms including $\theta_n(2\tau, 2m_l)$ can be reorganized using (A.18) and (A.19), in terms of $\theta_n(\tau, m_l)$ and $\theta_n(\tau, 0)$. So finally, N^{E_8} is written as a polynomial of $\theta_n(\tau, m_l)$, $\theta_n(\tau, \epsilon)$, with coefficients given by $\theta_n(\tau, 0)$.

Finally, to straightforwardly compare N and N^{E_8}, we want to express $\theta_n(\epsilon)$'s in terms of $\theta_n(\epsilon/2)$'s. Plugging $b = \frac{\epsilon}{2}$ and $a = \frac{\epsilon}{2} + \frac{p}{2}$ (with $p = 0, 1, \tau, \tau+1$) into (A.14) and (A.15), one obtains the desired formulae. Then inserting them into N, N^{E_8}, we obtain polynomials of $\theta_n(\tau, m_l)$, $\theta_n(\tau, \frac{\epsilon}{2})$ with coefficients given by $\theta_n(\tau, 0)$. Now we can evaluate $N^{E_8} - N$ on computer, by eliminating $\theta_1(m_l)$, $\theta_1(\epsilon/2)$, $\theta_2(0)$ by using (A.13). Then one finds $N^{E_8} - N = 0$, proving the equivalence of (3.25) and (3.26).

3 and 4 E-strings. We compare our elliptic genera (3.39) and (3.57) against the known results summarized in appendix B. The free energy is expanded as

$$F = \log Z = \sum_{n_b=1}^{\infty} w^{n_b} F_{n_b} = \sum_{n,g,n_b} (\epsilon_1 + \epsilon_2)^{2n} (\epsilon_1 \epsilon_2)^{g-1} w^{n_b} F^{(n,g,n_b)}\,, \tag{C.5}$$

where $F_1 = Z_1$, $F_2 = Z_2 - \frac{1}{2}Z_1^2$, $F_3 = Z_3 - Z_1 Z_2 + \frac{1}{3}Z_1^3$ and $F_4 = Z_4 - Z_1 Z_3 - \frac{1}{2}Z_2^2 + Z_1^2 Z_2 - \frac{1}{4}Z_1^4$. The coefficients $F^{(n,g,n_b)}$ computed from topological strings, summarized in appendix B, depend on η, E_2, E_4, E_6. Using (A.12), these can be arranged into expressions involving E_2 and $\theta_n(0)$ only.

On the other hand, if we set $m_l = 0$ and compute $F^{(n,g,n_b)}$ from our gauge theory indices, they will be rational functions of $\theta_n(0)$, η, $\theta_n^{(k)}(0)$. The derivatives $\theta_n^{(k)}(0)$ appear because we are expanding the index with ϵ_1, ϵ_2. We want to express our gauge theory expressions for $F^{(n,g,n_b)}$ in terms of $\theta_n(0)$'s and E_2 only, to compare with the results summarized in appendix B. Firstly, (A.12) can be used to eliminate η. The remaining task is to write $\theta_{1,2,3,4}^{(k)}(0)$ in terms of $\theta_n(0)$'s and E_2, which can be done in the following way.

Starting from the lowest non-vanishing derivatives (A.23) at $z = 0$, we can iteratively obtain $\theta_n^{(k)}(0)$ for higher k's. For example,

$$\begin{aligned}(\partial_z)^3 \theta_1(\tau, z)|_{z=0} &= -8\pi^2 (\partial_z)(q\partial_q)\theta_1(\tau, z)|_{z=0} = -8\pi^2 (q\partial_q)(\partial_z \theta_1(\tau, z))|_{z=0} \\ &= -16\pi^3 (q\partial_q)\eta^3 = -2\pi^3 \eta^3 E_2\end{aligned} \tag{C.6}$$

where (A.22) and (A.21) are applied at the last step. If we look at another example,

$$\begin{aligned}(\partial_z)^4 \theta_2(\tau, z)|_{z=0} &= -8\pi^2 (\partial_z)^2 (q\partial_q)\theta_2(\tau, z)|_{z=0} = -8\pi^2 (q\partial_q)(\partial_z^2 \theta_2(\tau, z))|_{z=0} \\ &= \frac{8}{3}\pi^4\, q\partial_q[\theta_2(0) \cdot (E_2 + \theta_3(0)^4 + \theta_4(0)^4)] \\ &= \frac{1}{9}\pi^4 \theta_2(0) \left[\alpha_2^2 + 4\theta_3(0)^4 \alpha_3 + 4\theta_4(0)^4 \alpha_4 + \frac{1}{12}(E_2^2 - E_4)\right].\end{aligned} \tag{C.7}$$

for $\alpha_2 \equiv E_2 + \theta_3(0)^4 + \theta_4(0)^4$, $\alpha_3 \equiv E_2 + \theta_2(0)^4 - \theta_4(0)^4$, and $\alpha_4 \equiv E_2 - \theta_2(0)^4 - \theta_3(0)^4$. At the last step, we applied (A.22) and (A.20). Going to higher derivatives involves no more difficulty, and this way we can always express $F^{(n,g,n_b)}$ in terms of $\theta_n(0)$'s and E_2 only.

So we find two expressions for $F^{(n,g,n_b)}$, depending on $\theta_n(0)$'s and E_2 only, one from the topological string calculus and another from our gauge theories. In particular, we

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focus on the 3 and 4 E-strings, obtained by expanding (3.39), (3.57). We computed the differences of the two expressions for $F^{(0,0,3)}$, $F^{(0,1,3)}$, $F^{(1,0,3)}$, $F^{(0,0,4)}$, $F^{(0,1,4)}$, $F^{(0,2,4)}$ on computer, substituting $\theta_2(0)^4 = \theta_3(0)^4 - \theta_4(0)^4$, and found zero in all cases. Of course, further analytic tests can also be easily made on computer for higher genus results.

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