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Supersymmetric M5 brane theories on $\mathrm{R} \times \mathrm{CP}^2$

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Abstract: We propose 4 and 12 supersymmetric conformal Yang-Mills-Chern-Simons theories on $\mathrm{R} \times \mathrm{CP}^2$ as multiple representations of the theory on M5 branes. These theories are obtained by twisted Z_k modding and dimensional reduction of the 6d (2,0) superconformal field theory on $\mathrm{R} \times \mathrm{S}^5$ and have a discrete coupling constant $\frac{1}{g_{YM}^2} = \frac{k}{4\pi^2}$ with natural number k. Instantons in these theories are expected to represent the Kaluza-Klein modes. For the $k = 1, 2$ cases, we argue that the number of supersymmetries in our theories should be enhanced to 32 and 16, respectively. For the $k = 3$ case, only the 4 supersymmetric theory gets the supersymmetric enhancement to 8. For the 4 supersymmetric case, the vacuum structure becomes more complicated as there are degenerate supersymmetric vacua characterized by fuzzy spheres. We calculate the perturbative part of the SU(N) gauge group Euclidean path integral for the index function at the symmetric phase of the 4 supersymmetric case and confirm it with the known half-BPS index. From the similar twisted Z_k modding of the $\mathrm{AdS}_7 \times \mathrm{S}^4$ geometry, we speculate that the M region is for $k \lesssim N^{1/3}$ and the type IIA region is $N^{1/3} \lesssim k \lesssim N$. When nonperturbative corrections are included, our theories are expected to produce the full index of the 6d (2,0) theory.

Keywords: Brane Dynamics in Gauge Theories, Conformal Field Models in String Theory, M-Theory

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Contents

1 Introduction

The physics of M5 branes [1–3] remains as one of great mysteries in M-theory [1, 2, 4]. Some fundamental structures of the underlying 6d (2,0) superconformal theory are not yet known. One promising approach is to study the 5d maximally supersymmetric gauge theory whose instantons may provide all Kaluza-Klein physics of the circle-compactified 6d theory [5, 6]. The study of a 1/4 BPS sector by the index calculation in this setting has provided the exact results on the DLCQ limit of the 6d (2,0) theory [7, 8]. However, one wants to have more devices to probe this 6d theory which allow, for example, the calculation of the full index function on $\mathrm{S}^1 \times \mathrm{S}^5$.

In this work we propose one such tool. First we put the 6d (2,0) theory on $\mathrm{R} \times \mathrm{S}^5$. The five sphere S^5 is a circle fibration over CP^2, and we mod out the theory by Z_k along this circle fiber with some additional twisting along a U(1) subgroup of the $\mathrm{Sp}(2)_R = \mathrm{SO}(5)$ R-symmetry. This allows a consistent truncation of the 6d theory to a 5d theory on $\mathrm{R} \times \mathrm{CP}^2$ with partially conserved supersymmetries. While we do not know the exact nonabelian 6d (2,0) theory, one can find this 5d nonabelian theory explicitly. The 5d theory has both Yang-Mills Chern-Simons terms and the Myers term for the scalar field. The Chern-Simons term is not the standard 5d Chern-Simons term but is of type $JAdA$ where J is the Kähler form on CP^2. This Z_k modding and dimensional reduction lead to the overall coupling constant $1/g_{YM}^2 = k/4\pi^2 r$ with the S^5 radius r, and so the 5d theories with $\mathrm{SU}(N)$ gauge group have a weakly coupled regime with the small 't Hooft coupling constant $\lambda = N/k$ when $k \gg N$. A different choice of twisting leads to a different 5d theory, even with different amount of supersymmetries. Here we construct two such 5d theories with either 4 or 12 supersymmetries.

The Killing spinors of S^5 can be singlet or triplet under the isometry group SU(3) of CP^2. We will show that the 4 and 12 supersymmetric theories have singlet and triplet Killing spinors, respectively. Our 5d theories do not appear in the standard classification of the super conformal field theories as Poincare supersymmetry is partially broken here [20, 21]. The supersymmetry on $R \times CP^2$ inherits the original superconformal symmetry and the eigenvalues of our Hamiltonian can be identified with the conformal dimension of an operator corresponding to the eigenstate on CP^2. As we have modded out some sector of the original theory, the quantum states of our theories for $k > 1$ have fewer quantum states than the original 6d theory.

As instantons in the 5d maximally supersymmetric Yang-Mills theory obtained by the dimensional reduction of the 6d (2,0) theory on $R^5 \times S^1$ are supposed to provide the Kaluza-Klein modes [9, 10], instantons in our theories are assumed also to provide all of the KK physics. Our theory has only one coupling constant which is discrete and quantized. In addition we expect that for $k = 1$ the amount of the supersymmetries of our theories should be enhanced to 32 as there is no modding. Especially, our theories with $k = 1$ might capture the all physics of the 6d (2,0) thoeory on $R \times S^5$ when one includes the Kaluza-Klein physics. Here, we restrict our interest to supersymmetric observables in both 5d and 6d theories. The supersymmetric observables in our theories are not sensitive to the subtle UV physics that usually arises from 5d non-renormalizable theories, and they can be exactly calculable involving non-perturbative sector. We believe these observables in our theories exactly agree with those of 6d (2,0) theory at $k = 1$. We will argue later on that for $k = 2$ case the number of supersymmetries of our 5d theories should be enhanced to 16. For $k = 3$ case, only the 4 supersymmetric theory gets the supersymmetric enhancement to 8. For $k \geq 4$ case we do not expect any enhancement.

The supersymmetry of our 5d theories is a part of 6d superconformal symmetry and so allows the definition of the superconformal indices in the 6d sense. Here we calculate the conformal index in the large k or free theory limit for the 4 supersymmetric theory and found that it matches exactly to what is expected.

There are three possibilities for our theory in the ultraviolet region: (1) UV finite and complete, (2) UV non-finite but renormalizable, and (3) UV non-finite and non-renormalizable. Unlike 5d SYM on R^{1+4} which has the dimensionful coupling with no additional adjustable parameter, our theory has the unique discrete quantized coupling constant with Chern-Simons term and also the weak coupling regime, and so has a better chance to be UV finite. It would be fascinating to figure out whether this is the case.

The $AdS_7 \times S^4$ geometry for the large N M5 branes is known [11] and a similar Z_k modding of this geometry would lead to the geometry corresponding to our theories. We speculate that there are three regions of k: the M-theory region for $1 \leq k \lesssim N^{1/3}$, the type IIA region for $N^{1/3} \lesssim k \lesssim N$, and the high curvature region for $N \lesssim k$. Such division is not concrete as the 11d circle radius, that is the dilation field, diverges at the boundary, which is the UV region of the field theory. These regions could be meaningful in the interior region of the AdS_7 space.

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Our approach is inspired in part by the ABJM theory on M2 branes which has Z_k modding of the SO(8) R-symmetry [12]. However, our Z_k modding is acting on both the space S^5 and the R^2 part of the scalar field space R^5. Our 5d theories are defined on a compact space CP^2 with SU(3) isometry instead of non-compact R^4.

We note that 5d JFA type Chern-Simons term has appeared in refs. [13, 14] while their setting is different from ours. There is another work by one of us (HK) and Seok Kim where the index on M5 brane has been approached by the 5d Yang-Mills theory on S^5 [15]. Not only a perturbative calculation on S^5 is done explicitly there but also a conjecture on instanton part has been provided. More relevant for the future work would be the index calculation on $\mathrm{S}^1 \times \mathrm{S}^4$ done recently for the 5d superconformal field theories [16]. There are some related recent works [17–19] on the supersymmetric theories on S^5.

A Myers' term appears in our 5d theories. For 4 supersymmetric case with SU(N) gauge group, one can have degenerate vacua which are characterized by supersymmetric fuzzy spheres and the partition of N. This classical degeneracy of vacua could be regarded as a blow up of D4 world volume from CP^2 to $\mathrm{CP}^2 \times \mathrm{S}^2$ and thus implies that some D6 brane giant gravitons contribute to the index. Such possibility raises many interesting questions which we would leave as a future problem.

All the fields in our 5d theories belong to the adjoint representation of the gauge group and the overall coupling constant is given as $1/g_{YM}^2 = k/4\pi^2 r$ with natural number k and the S^5 radius r. These 5d theories, obtained after the Z_k modding and the dimensional reduction, have the weak coupling limit in large k and discrete coupling constants beside the dimensionful factor r. The space CP^2 is compact and so the energy spectrum is expected to be discrete. The effective dimensionless coupling constant could be then discrete also. Our 5d theories might be ultraviolet complete when nonperturbative part is included. But our theories are not defined on flat R^{1+4} with Lorentz symmetry and so the usual perturbative expansion in momentum space is not available. The large k limit is the weak coupling limit and the $k = 1$ limit is the strong coupling limit. For the SU(N) or U(N) theory, there is also 't Hooft coupling constant $\lambda = N/k$. For large 't Hooft coupling limit the corresponding AdS geometry is obtained by the Z_k modding of the $\mathrm{AdS}_7 \times \mathrm{S}^4$ and somewhat complicated as the boundary geometry is a Z_k modding of the boundary geometry $\mathrm{R} \times \mathrm{S}^5 \times \mathrm{S}^4$.

The index function for a conformal field theory is an important tool to explore the theory [22–25]. The index function of the 6d (2,0) theory on $\mathrm{S}^1 \times \mathrm{S}^5$ is one of the major interests. The index for the U(1) theory on a single M5 has been done [26]. Our 5d theories have both perturbative parts and instanton parts. In this work, we restrict ourself to just the perturbative part of 4-supersymmetric case and find it to match with the known 1/2 BPS index on the single M5 brane [27].

The outline of this work is as follows. In section2 we start with the 6d abelian (2,0) theory and do the twisted Z_k modding and the dimensional reduction to obtain some supersymmetric 5d Yang-Mills Chern-Simons theories on $\mathrm{R} \times \mathrm{CP}^2$. In section3 we explore the properties of these theories, including the spectrum of the abelian theory. In section4 we introduce the index function and calculate it by the Euclidean path integral in the weak coupling limit. In section5 we conclude with some remarks. In appendices we include the properties of manifolds S^5 and CP^2 and the Killing spinors on them.

2 5d supersymmetric theories on $\mathrm{R} \times \mathrm{CP}^2$

Let us start with the 6d abelian (2,0) theory on R^{1+5} for the field $B_{MN}, \lambda, \phi_I (I = 1,2,3,4,5)$. The 3-form field strength $H = dB$ should be selfdual $H = {}^*H$. We start with the supersymmetric action with additional spectator field $H = -{}^*H$ which does not get involved in the supersymmetric transformation [28]. The bosonic part of the superconformal symmetry $\mathrm{OSp}(2,6|2)$ is made of the $\mathrm{SO}(2,6)$ conformal symmetry and $\mathrm{Sp}(2)_R = \mathrm{SO}(5)$ R-symmetry. The conformal dimensions of H, λ, ϕ_I are $3, 5/2, 2$, respectively. One does the radial quantization of the theory and obtains the Lorentz signature action on $\mathrm{R} \times \mathrm{S}^5$. The Cartan elements of the spatial rotation algebra $\mathrm{SU}(4) = \mathrm{SO}(6)$ are made of j_1, j_2, j_3 and the Cartan elements of the Lie algebra $\mathrm{Sp}(2)_R = \mathrm{SO}(5)$ are made of R_1, R_2. The R_1 rotates the scalar fields ϕ_1, ϕ_2 and R_2 rotates ϕ_4, ϕ_5. Spinor field λ belongs to $\mathbf{4}$ of $\mathrm{SU}(4)$ and $\mathbf{4}$ of $\mathrm{Sp}(2)_R$. Both the fermion field λ and supercharge Q transform identically under $\mathrm{SU}(4)$ and $\mathrm{Sp}(2)_R$. In terms of roots $\pm e_i \pm e_j, (i,j = 1,2,3)$ of $\mathrm{SO}(6)$, the spinor representations $\mathbf{4}$ and $\bar{\mathbf{4}}$ of $\mathrm{SU}(4)$ are given by the weights $(\pm e_1 \pm e_2 \pm e_3)/2$ with odd and even numbers of minus signs, respectively.

Let us do the radial quantization of the (2,0) theory on $\mathrm{R} \times \mathrm{S}^5$. See the appendix A for the metrics on S^5 and CP^2. The action on $R \times S^5$ is

$$S = \int_{R\times S^5} d^6x\sqrt{g}\,\Big\{ -\frac{1}{12}H_{MNP}H^{MNP} - \frac{i}{2}\bar{\lambda}\Gamma^M\hat{\nabla}_M\lambda - \frac{1}{2}\partial_M\phi_I\partial^M\phi_I - \frac{2}{r^2}\phi_I\phi_I\Big\}. \tag{2.1}$$

Here $\hat{\nabla}_M$ is the spinor covariant derivative on $\mathrm{R}\times\mathrm{S}^5$. From now on we put the S^5 radius r to be unity for the simplicity. The supersymmetric transformation for the tensor multiplet is

$$\begin{aligned}
\delta B_{MN} &= -\bar{\lambda}\Gamma_{MN}\epsilon = -\bar{\epsilon}\Gamma_{MN}\lambda,\\
\delta\phi_I &= -\bar{\lambda}\rho_I\epsilon = +\bar{\epsilon}\rho_I\lambda,\\
\delta\lambda &= +\frac{i}{6}H_{MNP}\Gamma^{MNP}\epsilon + i\partial_M\phi_I\Gamma^M\rho_I\epsilon - 2\phi_I\rho_I\tilde{\epsilon},\\
\delta\bar{\lambda} &= -\frac{i}{6}H_{MNP}\bar{\epsilon}\Gamma^{MNP} + i\partial_M\phi_I\bar{\epsilon}\Gamma^M\rho_I - 2\bar{\tilde{\epsilon}}\rho_I\phi_I.
\end{aligned} \tag{2.2}$$

The Killing spinors ϵ should satisfy

$$\hat{\nabla}_M\epsilon = \frac{i}{2r}\Gamma_M\tilde{\epsilon}, \quad \Gamma^M\hat{\nabla}_M\tilde{\epsilon} = 2i\epsilon, \tag{2.3}$$

which can be partially solved by $\tilde{\epsilon} = \pm\Gamma_0\epsilon$.

Note that

$$H_{MNP}\Gamma^{MNP}\epsilon = \frac{1}{2}(H_{MNP} + {}^*H_{MNP})\Gamma^{MNP}\epsilon, \tag{2.4}$$

where

$${}^*H_{MNP} = \frac{1}{6}\epsilon_{MNPQRS}H^{QRS}, \quad \epsilon_{0123456} = -1. \tag{2.5}$$

Only the selfdual part $H = {}^*H$ appears in the supersymmetry transformation. Thus the anti-selfdual part of the field strength transform as

$$\delta(H_{MNP} - {}^*H_{MNP}) = i\bar{\epsilon}\Gamma_{MNP}\Gamma^Q\partial_Q\lambda, \tag{2.6}$$

which vanishes on-shell.

The metric for the five sphere is

$$ds^2_{S^5} = ds^2_{\mathtt{CP}^2} + (dy + V)^2, \tag{2.7}$$

where $y \sim y + 2\pi$. The Kähler form J is given by

$$J = \frac{1}{2} dV. \tag{2.8}$$

We want to a Z_k modding of the 6d (2,0) theory along the fiber direction with identification

$$y \sim y + \frac{2\pi}{k}. \tag{2.9}$$

The Killing spinors on S^5 as shown in appendix B have nontrivial y-dependence and the above Z_k modding would remove them unless one introduces an additional twisting along some direction of $\mathrm{Sp}(2)_R = \mathrm{SO}(5)$ R-symmetry. Let us consider the plus sign case $\tilde{\epsilon} = +\Gamma_0 \epsilon$. With the notation for the eigenspinors $\gamma^{12}\epsilon^{s_1 s_2} = i s_1 \epsilon^{s_1 s_2}, \gamma^{34}\epsilon^{s_1 s_2} = i s_2 \epsilon^{s_1 s_2}$, we group the 16 Killing spinors ϵ_+ to the 4 and 12 spinors. The first group of the Killing spinors is made of

$$(\mathbf{I})\ \epsilon_+ \sim e^{-\frac{i}{2}t + \frac{3i}{2}y} \epsilon_0^{++}, \tag{2.10}$$

with constant spinors ϵ_0^{++} which form a singlet of SU(3) isometry of CP^2 and the fundamental representation of $\mathrm{Sp}(2)_R = \mathrm{SO}(5)$. The second group of the three independent Killing spinors is made of

$$(\mathbf{II})\ \epsilon_+ \sim e^{-\frac{i}{2}t - \frac{i}{2}y} (\epsilon_1^{+-}, \epsilon_2^{-+}, \epsilon_3^{--}), \tag{2.11}$$

where these are complicated CP^2-dependent matrix linear combinations of three constant spinors. They form a triplet of SU(3) isometry of CP^2 and the fundamental representation of $\mathrm{Sp}(2)_R = \mathrm{SO}(5)$. The exact form is not important here.

We want to cancel the y-dependence of the spinor parameter by introducing a twisting of the spinor parameter along the R-symmetry direction. There are many inequivalent and less supersymmetric choices. Here we focus on two choices for the simplicity and twist both spinor and scalar fields to be consistent with the supersymmetric transformation.

The first choice is to introduce new variables

$$(\mathbf{I})\ \ \epsilon_{\rm old} = e^{-\frac{3\rho_{45}}{2}y} \epsilon_{\rm new},\ \ \lambda_{\rm old} = e^{-\frac{3\rho_{45}}{2}y} \lambda_{\rm new},\ \ (\phi_4 + i\phi_5)_{\rm old} = e^{+3iy} (\phi_4 + i\phi_5)_{\rm new}. \tag{2.12}$$

This change of variables leads to

$$(\mathbf{I})\ \ \partial_y \to \partial_y + 3iR_2 \tag{2.13}$$

on new variables. Here R_2 is one of the Cartans of $\mathrm{Sp}(2)_R = \mathrm{SO}(5)$ R-symmetry. The corresponding $\mathrm{U}(1)_{R_2}$ transformation is $\phi_4 + i\phi_5 \to e^{i\alpha}(\phi_4 + i\phi_5)$ and $\lambda \to e^{-\frac{\rho_{45}}{2}\alpha}\lambda$. Note that the y-charges for all fermions are shifted by half-integers. Before the twisting, the fermions originally have half-integers y-charges and thus they are anti-periodic under $y \to$

$y+2\pi$. So the original fermionic modes in 6d (2,0) theory are all projected out by Z_k modding. The twisting provides to the fermionic modes extra half-integer shifts of y-charges (new fermions all become periodic under $y \to y+2\pi$), that makes some of fermionic modes survive under the Z_k quotient. The new spinor parameter has the y-dependence as $e^{3(i+\rho_{45})y/2}\epsilon_0^{++}$, and so we choose the constant spinor to satisfy $\rho_{45}\epsilon_0 = -i\epsilon_0$ to remove the y-dependence. This supersymmetry would survive under the Z_k modding. The possible R_1-charge of allowed ϵ_+ spinors is the eigenvalues $\pm i$ of ρ_{12}. As ϵ_- is a complex conjugate, there would be four surviving supersymmetries in the first case after Z_k modding.

The second one is to introduce new variables so that

$$(\mathbf{II}) \quad \epsilon_{\rm old} = e^{+\frac{\rho_{45}}{2}y}\epsilon_{\rm new}, \;\; \lambda_{\rm old} = e^{+\frac{\rho_{45}}{2}y}\lambda_{\rm new}, \;\; (\phi_4+i\phi_5)_{\rm old} = e^{-iy}(\phi_4+i\phi_5)_{\rm new}. \tag{2.14}$$

This leads to the change of the derivative ∂_y on new fields to

$$(\mathbf{II}) \;\; \partial_y \to \partial_y - iR_2. \tag{2.15}$$

The y-dependence of the Killing spinors ϵ_+ can be removed once $\rho_{45}\epsilon_+ = -i\epsilon_+$. These two triplets of Killing spinors ϵ_+ with eigenvalue of $\rho_{12} = \pm i$ would survive under the Z_k modding and so the resulting theory would have 12 supersymmetries. One can rewrite 6d action using these new variables and see that the 6d kinetic terms provide extra mass terms to the R_2-charged fields, $\phi_{i=4,5}$, λ, such as

$$\begin{aligned}
(\mathbf{I}) \;\; -\frac{1}{2}\partial_M\phi_i\partial^M\phi_i - \frac{i}{2}\bar\lambda\Gamma^M\hat\nabla_M\lambda &\to -\frac{1}{2}\partial_M\phi_i\partial^M\phi_i - \frac{9}{2r^2}(\phi_i)^2 - \frac{i}{2}\bar\lambda\Gamma^M\hat\nabla_M\lambda + \frac{3i}{4r}\bar\lambda\Gamma^5\rho_{45}\lambda\,, \\
(\mathbf{II}) \;\; -\frac{1}{2}\partial_M\phi_i\partial^M\phi_i - \frac{i}{2}\bar\lambda\Gamma^M\hat\nabla_M\lambda &\to -\frac{1}{2}\partial_M\phi_i\partial^M\phi_i - \frac{1}{2r^2}(\phi_i)^2 - \frac{i}{2}\bar\lambda\Gamma^M\hat\nabla_M\lambda - \frac{i}{4r}\bar\lambda\Gamma^5\rho_{45}\lambda
\end{aligned} \tag{2.16}$$

The Z_k modding of the new spinor and scalar fields is given to be

$$\begin{aligned}
(\mathbf{I}) \; \lambda(y)_{\rm old} &\sim e^{-\frac{3\pi\rho_{45}}{k}}\lambda\left(y+\frac{2\pi}{k}\right)_{\rm old}, \quad (\phi_4+i\phi_5)(y)_{\rm old} \sim e^{+\frac{6\pi i}{k}}(\phi_4+i\phi_5)\left(y+\frac{2\pi}{k}\right)_{\rm old}, \\
(\mathbf{II}) \; \lambda(y)_{\rm old} &\sim e^{+\frac{\pi\rho_{45}}{k}}\lambda\left(y+\frac{2\pi}{k}\right)_{\rm old}, \quad (\phi_4+i\phi_5)(y)_{\rm old} \sim e^{-\frac{2\pi i}{k}}(\phi_4+i\phi_5)\left(y+\frac{2\pi}{k}\right)_{\rm old}.
\end{aligned} \tag{2.17}$$

Such consistent Z_k modding of the 6d (2,0) theory reduces the degrees of freedom and the number of supersymmetries. Still we do not know the exact form of the resulting 6d theory.

Let us now do the dimensional reduction of the 6d (2,0) abelian theory to 5d by requiring the new field variables to be independent of y. Then the y-independent new spinor and scalar fields are invariant under the Z_k modding and so are allowed. For the two-form tensor field, we can either do the dimensional reduction with identification $H_{\mu\nu5} \sim F_{\mu\nu}$, or find the gauge kinetic term by the supersymmetry completion. As we found the y independent Killing spinors for the superconformal transformation, the supersymmetric transformation under these Killing spinors will not introduce additional y dependence between the fields. This Z_k modding in large k limit would shrink the circle fiber size relative to $\mathrm{R}\times\mathrm{CP}^2$ size and so the theory would become more close to the 5d theory.

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We first consider only the action for the scalar and spinor fields and then fix the the gauge kinetic term to complete the supersymmetry for the abelian case. Then one generalizes the theory to the nonabelian case. Only subtlety is the right normalization for the coupling constant. For both cases we argue in the next section that the 5d gauge coupling constant is given by

$$\frac{1}{g_{YM}^2} = \frac{k}{4\pi^2 r}, \tag{2.18}$$

where r is the radius of the S^5 sphere and is regard as unity as it is only length scale of the theory. Unlike the physics on $R^{1+4} \times S^1$, the energy spectrum E on the compact space CP^2 to be discrete and $E \sim n/r$ and so the dimensionless coupling constant $g_{YM}^2 E$ to be discrete also. The theory becomes weakly coupled in large k limit. As the fields are in the adjoint representation of the gauge group, one expects the presence of 't Hooft coupling constant

$$\frac{N}{k} \tag{2.19}$$

for $\mathrm{U}(N)$ gauge theory.

To be more concrete, we use the four-component notation for the 5d spinors as given in appendix A. The symplectic reality condition becomes

$$\lambda = BC\lambda^*, \ \epsilon = BC\epsilon^*. \tag{2.20}$$

The Killing spinor equation for the y-independent new spinor parameter ϵ_{new} becomes

$$\partial_t \epsilon = \tfrac{i}{2}\gamma_0 \tilde{\epsilon}, \ D_m \epsilon = -\tfrac{i}{2} J_{mn}\gamma^n \epsilon + \tfrac{i}{2}\gamma_m \tilde{\epsilon}, \tag{2.21}$$

where $m = 1, 2, 3, 4$. These spinor variables satisfy additional conditions for two cases:

$$\begin{aligned}
&(\mathbf{I}) \ \ \rho_{45}\epsilon_+ = -i\epsilon_+, \quad D_m = \nabla_m + \frac{3\rho_{45}}{2}V_m, \quad \tilde{\epsilon} = -\left[3\rho_{45} + \frac{1}{2}J_{mn}\gamma^{mn}\right]\epsilon, \\
&(\mathbf{II}) \ \ \rho_{45}\epsilon_+ = -i\epsilon_+, \quad D_m = \nabla_m - \frac{\rho_{45}}{2}V_m, \quad \tilde{\epsilon} = \left[\rho_{45} - \frac{1}{2}J_{mn}\gamma^{mn}\right]\epsilon,
\end{aligned} \tag{2.22}$$

where ∇_m is the spinor covariant derivative on CP^2. The $\mathrm{U}(1)$ rotation by R_2 is deformed to a $\mathrm{U}(1)'_R$ due to the twisting for both cases.

The resulting 4 supersymmetric nonabelian 5d action on $\mathrm{R} \times \mathrm{CP}^2$ for the first case is

$$\begin{aligned}
S_{\mathbf{I}} = \frac{k}{4\pi^2}\int_{\mathrm{R}\times\mathrm{CP}^2} d^5x\sqrt{|g|}\ \mathrm{tr}\Big[&-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2\sqrt{|g|}}\epsilon^{\mu\nu\rho\sigma\eta}J_{\mu\nu}\Big(A_\rho\partial_\sigma A_\eta - \frac{2i}{3}A_\rho A_\sigma A_\eta\Big) \\
&- \frac{1}{2}D_\mu\phi_I D^\mu\phi_I + \frac{1}{4}[\phi_I,\phi_J]^2 - i\epsilon_{abc}\phi_a[\phi_b,\phi_c] - 2\phi_a^2 - \frac{13}{2}\phi_i^2 \\
&- \frac{i}{2}\bar{\lambda}\gamma^\mu D_\mu\lambda - \frac{i}{2}\bar{\lambda}\rho_I[\phi_I,\lambda] - \frac{1}{8}\bar{\lambda}\gamma^{mn}\lambda J_{mn} + \frac{3}{4}\bar{\lambda}\rho_{45}\lambda\Big],
\end{aligned} \tag{2.23}$$

where $I=1,2,3,4,5$, $a=1,2,3$, $i=4,5$ and

$$\begin{aligned}
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu],\\
D_\mu \phi_a &= \partial_\mu \phi_a - i[A_\mu, \phi_a],\\
D_\mu \phi_i &= \partial_\mu \phi_i - i[A_\mu, \phi_i] + 3V_\mu \epsilon_{ij}\phi_j,\\
D_\mu \lambda &= \left[\partial_\mu + \frac{1}{4}\omega_\mu^{\rho\sigma}\gamma_{\rho\sigma} + \frac{3}{2}V_\mu \rho_{45}\right]\lambda - i[A_\mu, \lambda].
\end{aligned} \tag{2.24}$$

Note that $J_{0m}=0$ and J_{mn} is the Kähler 2-form on CP^2. The supersymmetric transformation is

$$\begin{aligned}
\delta A_\mu &= +i\bar\lambda\gamma_\mu\epsilon = -i\bar\epsilon\gamma_\mu\lambda,\\
\delta\phi_I &= -\bar\lambda\rho_I\epsilon = \bar\epsilon\rho_I\lambda,\\
\delta\lambda &= +\frac{1}{2}F_{\mu\nu}\gamma^{\mu\nu}\epsilon + iD_\mu\phi_I\rho_I\gamma^\mu\epsilon - \frac{i}{2}[\phi_I,\phi_J]\rho_{IJ}\epsilon - 3\epsilon_{ij}\phi_i\rho_j\epsilon - 2\phi_I\rho_I\tilde\epsilon.
\end{aligned} \tag{2.25}$$

The supercharge Q is a singlet under SU(3) isometry of CP^2 and a doublet under $\mathrm{SU}(2)_R$ with nontrivial $\mathrm{U}(1)'_R$ charge. Thus the supergroup behind the first superconformal model would be SU(1|2).

The 12 supersymmetric 5d action on $\mathrm{R}\times\mathrm{CP}^2$ for the second case is

$$\begin{aligned}
S_{\mathbf{II}} = \frac{k}{4\pi^2}\int_{\mathrm{R}\times\mathrm{CP}^2} d^5x\sqrt{|g|}\ \mathrm{tr}\Big[&-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2\sqrt{|g|}}\epsilon^{\mu\nu\rho\sigma\eta}J_{\mu\nu}\Big(A_\rho\partial_\sigma A_\eta - \frac{2i}{3}A_\rho A_\sigma A_\eta\Big)\\
&-\frac{1}{2}D_\mu\phi_I D^\mu\phi_I + \frac{1}{4}[\phi_I,\phi_J]^2 + \frac{i}{3}\epsilon_{abc}\phi_a[\phi_b,\phi_c] - 2\phi_a^2 - \frac{5}{2}\phi_i^2\\
&-\frac{i}{2}\bar\lambda\gamma^\mu D_\mu\lambda - \frac{i}{2}\bar\lambda\rho_I[\phi_I,\lambda] - \frac{1}{8}\bar\lambda\gamma^{mn}\lambda J_{mn} - \frac{1}{4}\bar\lambda\rho_{45}\lambda\Big],
\end{aligned} \tag{2.26}$$

where $I=1,2,3,4,5$, $a=1,2,3$, $i=4,5$ and

$$\begin{aligned}
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu],\\
D_\mu \phi_a &= \partial_\mu \phi_a - i[A_\mu, \phi_a],\\
D_\mu \phi_i &= \partial_\mu \phi_i - i[A_\mu, \phi_i] - V_\mu \epsilon_{ij}\phi_j,\\
D_\mu \lambda &= \left[\partial_\mu + \frac{1}{4}\omega_\mu^{\rho\sigma}\gamma_{\rho\sigma} - \frac{1}{2}V_\mu \rho_{45}\right]\lambda - i[A_\mu, \lambda].
\end{aligned} \tag{2.27}$$

The supersymmetric transformation is

$$\begin{aligned}
\delta A_\mu &= i\bar\lambda\gamma_\mu\epsilon = -i\bar\epsilon\gamma_\mu\lambda,\\
\delta\phi_I &= -\bar\lambda\rho_I\epsilon = \bar\epsilon\rho_I\lambda,\\
\delta\lambda &= +\frac{1}{2}F_{\mu\nu}\gamma^{\mu\nu}\epsilon + iD_\mu\phi_I\rho_I\gamma^\mu\epsilon - \frac{i}{2}[\phi_I,\phi_J]\rho_{IJ}\epsilon + \epsilon_{ij}\phi_i\rho_j\epsilon - 2\phi_I\rho_I\tilde\epsilon.
\end{aligned} \tag{2.28}$$

The supercharge Q is a triplet under SU(3) isometry of CP^2 and a doublet under $\mathrm{SU}(2)_R$ with nontrivial $\mathrm{U}(1)'_R$ charge. Thus the supergroup behind the second superconformal model would be SU(3|2).

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3 Properties of the 5d theories

There are several properties of these 5d theories we like to focus in this section. While we will not explore in detail in the present work, instantons in our theories would play the Kaluza-Klein modes for the circle fiber as the case in the maximally supersymmetric 5d Yang-Mills theory on R^5.

The instanton number on CP^2 is

$$\nu = \frac{1}{8\pi^2}\int_{\mathrm{CP}^2} \mathrm{Tr}(F\wedge F) = \frac{1}{16\pi^2}\int_{\mathrm{CP}^2} d^4x\sqrt{|g|}\ \mathrm{Tr}F_{\mu\nu}\tilde{F}^{\mu\nu}. \tag{3.1}$$

The eigenvalue of the Hamiltonian of the 6d (2,0) theory on $\mathrm{R}\times\mathrm{S}^5$ for an eigenstate is the conformal dimension of the corresponding operator. Similarly the Hamiltonian for our 5d theories would have the conformal dimensions as the eigenvalues. The abelian scalar field harmonics on S^5 is discussed in detail in later part of this section which shows that the lowest conformal dimension for the untwisted scalar field ϕ_a is two as expected. Upon Z_k modding, first nontrivial Kaluza-Klein modes start with conformal dimension $k+2$. Such KK modes along the circle fiber is supposed to be represented by instantons in the 5d theory. As a single instanton has mass $4\pi^2/g_{YM}^2$ with our normalization $1/(4g_{YM}^2)\mathrm{Tr}F^2$ where F is $N\times N$ hermitian matrix valued two-form for $\mathrm{U}(N)$ gauge group and the KK modes has the additional mass k, the inverse coupling coefficient $1/g_{YM}^2$ is chosen to be $k/4\pi^2$. Instantons on CP^2 have been explored in ref. [29]. It would be interesting to consider their work in our index calculation context.

While we do not have an argument for the quantization of the Chern-Simons term in the 3d sense, there is a simple argument in the 1d sense. Let us consider the abelian case with the spatial part of the vector potential $A=V$ so that $F=dA=2J$, which has half instanton number and has 2π flux on non-contractiable two cycles of CP^2. The Chern-Simons term in this background becomes

$$\frac{k}{4\pi^2}\int_{\mathrm{R}\times\mathrm{CP}^2} d^5x\ \frac{1}{2}\epsilon^{\mu\nu\rho\sigma\eta}J_{\mu\nu}\partial_\rho A_\sigma A_\eta\ \Rightarrow \int dt\ kA_0 \tag{3.2}$$

The Chern-Simons level k for this 1d $\mathrm{U}(1)$ theory is again integer quantized as expected.

The quadratic Chern-Simons term has been noted before [13, 14]. A beautiful argument in ref. [13] is that the y independent field equation for the 3-form tensor field on $R\times S^5$ leads naturally to the presence of the quadratic Chern-Simons term. Another argument is that the instantons are KK modes along the fiber direction and KK gauge field V_pdx^p is a gauge field on CP^2 space with magnetic field $2J$. Whenever the instanton moves, it feels the background magnetic field and so the interaction term should be proportional to V_pdx^p/dt where x^p is the position of a point like instanton on CP^2. The natural field theoretic expression is then the Chern-Simons term. It would be interesting to find more argument to support our choice of the coupling constant. The full effect of this Chern-Simons term is not clear at this moment.

The strongest coupling occurs for the $k=1$ case. For this value, there is no Z_k modding and so the supersymmetry should be enhanced to the maximal value 32 with $\mathrm{OSp}(8|2)$ supergroup. For larger k, the story is more complicated. Let us examine the

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6d Killing spinors after twisting but before dimensional reduction. Their dependence on the fiber direction y would be depending on the R_2 charge. For the four supersymmetric case (**I**), SU(3) singlet Killing spinors get split to $\epsilon_+ \sim e^{-it/2+0y}, e^{it/2+3iy}$ and SU(3) triplet Killing spinors get split to $\epsilon_+ \sim e^{-\frac{i}{2}t-2iy}, e^{-\frac{i}{2}t+iy}$ depending on R_2 charge. Upon the dimensional reduction, only y-independent modes and Killing spinors are realized explicitly. For the $k=1$ case, all KK modes would be realized non-perturbatively and so are nontrivial Killing spinors. For the $k=2$ case, the half of the triplet Killing spinors with even y momentum should be realized non-perturbatively in our 5d theory, and the total number of supersymmetries should be enhanced to 16. For the $k=3$ case, another half of the singlet Killing spinors can be realized non-perturbatively instead, and so the total number of supersymmetries would get enhanced to 8. For the $k \geq 4$, any supersymmetry enhancement is not expected. For the 12 supersymmetric case (**II**), SU(3) triplet Killing spinors get split to $\epsilon_+ \sim e^{-\frac{i}{2}t+0y}, e^{-\frac{i}{2}t-iy}$ and SU(3) singlet Killing spinors get split to $\epsilon_+ \sim e^{-\frac{i}{2}t+2iy}, e^{-\frac{i}{2}t+iy}$. For $k=2$ of this case, the half of singlet Killings spinors can be also realized and so the total supersymmetry would be 16. But there would be no more supersymmetry enhancement for $k \geq 3$.

One could ask whether instantons and anti-instantons without any other fields turned on are BPS in our 5d theories. Here we just consider the selfdual or anti-selfdual gauge field strength, leaving the study of the instanton solution itself to the next paper. For the first case (**I**), we note that $\gamma_{1234}\epsilon = -\epsilon$ and only anti-instantons can be BPS. For the second case (**II**), both instantons and anti-instantons can be BPS. The amount of preserved super symmetries is interesting also. For the first case the anti-instantons preserve all of 4 susy. For the second case the anti-instantons preserve 4 susy and instantons preserve 8 susy.

While the instanton mass fixes the coupling constant, its 6d field theoretic origin can be read as follows:

$$\begin{aligned} S_{6d} &= \int_{\mathrm{R}\times\mathrm{S}^5} d^6x\sqrt{g}\,\Big(-\frac{1}{2}\partial_\mu(\phi_I)_{6d}\partial^\mu(\phi_I)_{6d}+\cdots\Big) \\ \to S_{5d} &= \frac{2\pi r}{k}\int_{\mathrm{R}\times\mathrm{CP}^2} d^5x\sqrt{g}\,\Big(-\frac{1}{2}\partial_\mu(\phi_I)_{6d}\partial^\mu(\phi_I)_{6d}+\cdots\Big) \\ \to S_{5d} &= \frac{k}{4\pi^2 r}\int_{\mathrm{R}\times\mathrm{CP}^2} d^5x\sqrt{g}\,\Big(-\frac{1}{2}\partial_\mu(\phi_I)_{5d}\partial^\mu(\phi_I)_{5d}+\cdots\Big) \end{aligned} \tag{3.3}$$

with the mass dimension for the 6d field ϕ_I being two and the mass dimension for the final 5d field ϕ_I being one. The scalar field is rescaled so that

$$(\phi_I)_{5d} = \frac{2\pi\sqrt{2\pi r}}{k}(\phi_I)_{6d}. \tag{3.4}$$

There is a Meyer's term in the potential which leads to nontrivial vacuum structure only for the first case. The classical supersymmetric vacua are given by the fuzzy 2-spheres which are given by the vacuum equation

$$(\mathbf{I}) \quad -i[\phi_a,\phi_b] = 2\epsilon_{abc}\phi_c. \tag{3.5}$$

Naively, they would be D4 brane blown up to D6 brane whose world volume topology is $\mathrm{R}\times\mathrm{CP}^2\times\mathrm{S}^2$, and there would be corresponding giant graviton solutions. The exact nature of these vacua and their role will be explored in future.

The Gauss law in the U(1) theory implies

$$\frac{k}{4\pi^2}D_m F^{m0} + \frac{k}{4\pi^2}e^{0mnpq}J_{mn}F_{pq} = 0. \tag{3.6}$$

Total charge should be zero in the compact CP^2. As J is selfdual, the anti-selfdual flux

$$F \sim e^1 \wedge e^2 - e^3 \wedge e^4 \tag{3.7}$$

seems possible without violation of the Gauss law but it does not satisfy $dF = 0$. The selfdual configuration $F = 2J$ in the abelian theory with the instanton number one half is not allowed due to the Gauss law without excitation of other fields. In nonabelian theories, there could be other charged matter fields and so the Gauss law could be satisfied nontrivially. There may be some monopole-like operator as in 3d case [25].

While the second case has more supersymmetries, the first case is simpler as the Killing spinor is constant spinor on CP^2. We will focus on the first case from now on. Still, it is hard to penetrate the detail physics of the theory yet. We do not see any restriction on the gauge group unlike the 6d theory [1, 2].

Let us briefly mention the spectrum of the theory for first type (**I**) for abelian case. The detail spectrum for the scalar and fermion fields on S^5 is given in [15]. As the index calculation in the next section provides the detail of the spectrum on CP^2, here we just focus on the scalar field spectrum. The spectrum of a scalar field of conformal dimension 2 on $R \times S^5$ has the mass

$$(-\nabla^2_{S^5} + 4)Y^{\ell_1,\ell_2} = (\ell_1 + \ell_2 + 2)^2 Y^{\ell_1,\ell_2}, \; -i\partial_y Y^{\ell_1,\ell_2} = (\ell_1 - \ell_2)Y^{\ell_1,\ell_2}. \tag{3.8}$$

The highest weight of the given irreducible representation Y^{ℓ_1,ℓ_2} would be $\ell_1 w_1 + \ell_2 w_2$ with two fundamental weights w_1, w_2 of SU(3). The dimension of the representation of the highest weight (ℓ_1, ℓ_2) is $(\ell_1+1)(\ell_2+1)(\ell_1+\ell_2+2)/2$. For $\phi_{1,2,3}$ fields, there is no twisting. Z_k modding puts the constraints $\ell_1 - \ell_2 = kn$ with integer n. The y-independent mode with $Y^{\ell,\ell}$ has the spectrum on CP^2 as

$$(-\nabla^2_{\mathrm{CP}^2} + 4)Y^{\ell,\ell} = 4(\ell + 1)^2 Y^{\ell,\ell}, \tag{3.9}$$

with the degeneracy $(\ell+1)^3$. Note that the conformal dimension of this mode is $\varepsilon = 2\ell+2$ and so it starts from 2 as we expect for the scalar field in the 6d theory. The first KK mode would with either $(\ell_1, \ell_2) = (k, 0)$ or $(0, k)$. Both of them have the conformal dimension $\varepsilon = k + 2$ with degeneracy $(k + 1)(k + 2)/2$.

The twisted mode $\phi_4+i\phi_5$ has more complicated y-independent modes and KK modes. The y independent mode is given by $Y^{\ell,\ell+3}$ or $Y^{\ell+3,\ell}$ with conformal dimension $\varepsilon = 2\ell+5$. One can do the similar analysis for the fermion field whose conformal dimension on CP^2 starts from $\varepsilon = 5/2$ as expected for the 6d fermion. The vector field analysis done in next section shows that its conformal dimension on CP^2 starts from $\varepsilon = 4$ not 3 which is expected for the three-form tensor field in 6d. There may be no constant three form among the harmonics on S^5. Instantons with perturbative effects should reproduce the KK modes. We hope to come back to these issues near future.

4 Superconformal index

We will now define the superconformal index on 6d (2,0) theory and analyze its properties upon the Z_k modding introduced in section 2. Later we will relate this index with the 5d index on $\mathrm{R}\times\mathrm{CP}^2$. The full 6d index would be obtainable from 5d computation involving the non-perturbative instanton states. The superconformal index encodes the spectrum of BPS states of the radially quantized theory on $\mathrm{R}\times\mathrm{S}^5$. More precisely, the index we will define shortly counts the BPS states annihilated by a chosen supercharges Q and its conjugate S among 32 supercharges in 6d. The chosen supercharge Q satisfies the algebra

$$\{Q,S\}=\varepsilon-j_1-j_2-j_3+2R_1+2R_2\equiv\Delta, \tag{4.1}$$

and hence the index will count BPS states saturating the bound $\Delta=0$. Here the supercharge Q has charges as $j_1=j_2=j_3=-\frac{1}{2}, R_1=R_2=-\frac{1}{2}$.

The superconformal index of the (2,0) theory is defined as

$$I(x,y_1,y_2,q)=\mathrm{tr}\Big[(-1)^F x^{\varepsilon+R_1}y_1^{j_1-j_2}y_2^{j_2-j_3}q^j\Big], \tag{4.2}$$

where $x=e^{-\beta}, y_1=e^{-i\gamma_1}, y_2=e^{-i\gamma_2}$ denote the chemical potentials for the Cartan generators of the subalgebra commuting with Q, and $j=j_1+j_2+j_3-3R_2$. This index for a single M5-brane and its gravity dual theory at large N are studied in [26]. As the abelian (2,0) theory is free, one can easily compute the index for the single M5-brane theory by reading off BPS letters from the field content of the (2,0) theory. The index of the U(1) (2,0) theory is given by the Plethystic exponential of the single letter index f

$$\begin{aligned} I&=\exp\left[\sum_{n=1}^{\infty}\frac{1}{n}f(x^n,y_i^n,q^n)\right],\\ f(x,y_1,y_2,q)&=\frac{x+x^2q^3-x^2q^2(1/y_1+y_1/y_2+y_2)+x^3q^3}{(1-xqy_1)(1-xqy_2/y_1)(1-xq/y_2)}. \end{aligned} \tag{4.3}$$

The denominator comes from the derivatives, the first two terms of the numerator come from the scalar fields, three minus terms in the numerator come from the fermion fields, and the last term in the numerator comes from the fermion field equation. There is no contribution from the two-form tensor field. One interesting limit of this index is to take $q\to 0$ limit where the index reduces to the half-BPS index that is the index function of half-BPS states (preserving 16 supersymmetries). In this limit, the letter index simply becomes $f=x$ and it reflects that only a single complex scalar $\phi_1-i\phi_2$ contributes to the index. The A_{N-1} non-abelian version of the half-BPS index [27] is already given by

$$I_{1/2-\mathrm{BPS}}=\prod_{m=1}^{N}\frac{1}{1-x^m}. \tag{4.4}$$

This is the index we will reproduce in this section by calculating the perturbative part of the corresponding Euclidean path integral on $\mathrm{S}^1\times\mathrm{CP}^2$.

Now we turn to the Z_k modding of the superconformal index. We introduced in section 2 the Z_k quotient along the circular fiber direction y twisted by R_2 rotation. The

j corresponds to the rotation of this twisted y direction. The modding leaves only the Z_k singlet states carrying $j = kn$ ($n \in Z$) charges and truncates all other states. Accordingly, the index of the 6d theory with Z_k quotient is defined as

$$I_{Z_k} = \mathrm{tr}\Big[(-1)^F x^{\varepsilon+R_1} y_1^{j_1-j_2} y_2^{j_2-j_3} q^j\Big]\Big|_{j=kn}. \tag{4.5}$$

When $k = 1$, it reproduces the index for the (2,0) theory discussed above. On the other hand, at infinite k limit or zero coupling limit, all the KK states with non-zero j charge are truncated and the index reduces to the 5d index counting the BPS states of the free theory on $\mathrm{R} \times \mathrm{CP}^2$. This limit is achieved by taking $q \to 0$ limit in the index computation. Here, we note that this index at infinite k coincides with the half-BPS index (4.4) as two limits are achieved identically by $q \to 0$.

We expect that the 5d index including the non-perturbative instanton states can reproduce the full 6d superconformal index. The 5d theory of the first case (**I**) introduced in section 2 preserves the same supercharge Q used to define the 6d index, and, therefore, we can define the 5d index in the same way as the 6d index (4.2). The perturbative states in 5d theory correspond to the j singlet modes while the instanton states realize the KK states with non-zero j charge. We thus identify the instanton number with the KK momentum number j.

The index can be considered as the Euclidean path integral of the 5d theory on $\mathrm{S}^1 \times \mathrm{CP}^2$

$$I(x, y_i, q) = \int_{\mathrm{S}^1\times\mathrm{CP}^2} \mathcal{D}\Psi e^{-S^E_{\mathbf{I}}[\Psi]}. \tag{4.6}$$

The twisted boundary condition along the time circle S^1 of radius βr is considered. The Euclidean version of the action (2.23) is given by

$$\begin{aligned} S^E_{\mathbf{I}} = \frac{k}{4\pi^2 r}\int_{\mathrm{S}^1\times\mathrm{CP}^2} d^5x\sqrt{|g|}\ \mathrm{tr}\bigg[&\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\epsilon^{\mu\nu\lambda\rho\sigma}J_{\mu\nu}\left(A_\lambda\partial_\rho A_\sigma - \frac{2i}{3}A_\lambda A_\rho A_\sigma\right) \\ &+\frac{1}{2}D_\mu\phi_I D^\mu\phi_I - \frac{1}{4}[\phi_I,\phi_J]^2 + \frac{i}{r}\epsilon^{abc}[\phi_a,\phi_b]\phi_c + \frac{2}{r^2}(\phi_a)^2 + \frac{13}{2r^2}(\phi_i)^2 \\ &-\frac{i}{2}\lambda^\dagger\gamma^\mu D_\mu\lambda - \frac{i}{2}\lambda^\dagger\rho_I[\lambda,\phi_I] - \frac{1}{8r}\lambda^\dagger J_{\mu\nu}\gamma^{\mu\nu}\lambda + \frac{3}{4r}\lambda^\dagger\rho_{45}\lambda\bigg], \end{aligned} \tag{4.7}$$

where the fermion λ is subject to the reality condition $\lambda = BC\lambda^*$ and the radius r of S^5 is introduced again. The twisted boundary condition shifts the time derivative such as

$$\partial_\tau \to \partial_\tau - \frac{\beta}{\beta r}R_1 - \frac{i\gamma_1}{\beta r}(j_1 - j_2) - \frac{i\gamma_2}{\beta r}(j_2 - j_3), \tag{4.8}$$

and, from now on, we consider the time derivatives as this shifted one. The action is invariant under the supersymmetry transformation

$$\begin{aligned} \delta\phi_I &= -\lambda^\dagger\rho_I\epsilon, \\ \delta A_\mu &= -i\lambda^\dagger\gamma_\mu\epsilon, \\ \delta\lambda &= \frac{1}{2}F_{\mu\nu}\gamma^{\mu\nu}\epsilon - iD_\mu\phi_I\gamma^\mu\rho_I\epsilon - \frac{i}{2}[\phi_I,\phi_J]\rho_{IJ}\epsilon + \frac{3}{r}\epsilon_{ij}\phi_i\rho_j\epsilon - \frac{2i}{r}\phi_I\rho_I\tilde{\epsilon}. \end{aligned} \tag{4.9}$$

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The supersymmetry parameter ϵ satisfies the conditions

$$D_\mu\epsilon = -\frac{i}{2r}J_{\mu\nu}\gamma^\nu\epsilon + \frac{1}{2r}\gamma_\mu\tilde{\epsilon}, \quad \frac{3}{2}\rho^{45}\epsilon = -\frac{1}{4}J_{\mu\nu}\gamma^{\mu\nu}\epsilon + \frac{i}{2}\tilde{\epsilon}, \quad \tilde{\epsilon} = i\rho^{45}\gamma_\tau\epsilon, \tag{4.10}$$

and we found four solutions to these conditions,

$$\gamma_{12}\epsilon_+ = \gamma_{45}\epsilon_+ = -\rho^{45}\epsilon_+ = i\epsilon_+, \tag{4.11}$$

and its conjugation $\epsilon_- = BC\epsilon_+^*$. It turns out that the four Killing spinors are convariantly constant on CP^2

$$D_m\epsilon_\pm = 0 \ (m = 1,2,3,4). \tag{4.12}$$

We would like to evaluate the superconformal index using the localization technique. The localization would lead to the path integral over the instanton configuration on CP^2 base. We leave the calculation of the nonperturbative instanton contributions for future work.

At infinite k, the gaussian integral of the quadratic equations produces the exact result. For convenience, let us divide the field content to a vector multiplet and an adjoint hypermultiplet (though there is no notion of the hypermultiplet as the theory preserves only 4 supercharges). We first pick up a complex supercharge Q corresponding to $\rho_{12}\epsilon = -i\epsilon$ and decompose the spinors as

$$\epsilon = \begin{pmatrix}\epsilon_- \\ \epsilon_+\end{pmatrix} \otimes \begin{pmatrix}1 \\ 0\end{pmatrix}, \quad \lambda = \begin{pmatrix}\chi^1 \\ \chi^2\end{pmatrix} \otimes \begin{pmatrix}1 \\ 0\end{pmatrix} + \begin{pmatrix}\psi^1 \\ \psi^2\end{pmatrix} \otimes \begin{pmatrix}0 \\ 1\end{pmatrix}. \tag{4.13}$$

Then the vector multiplet consists of A_μ, χ, ϕ_3 and the hypetmultiplet consists of two complex scalar q^A and a complex fermion ψ defined as

$$q_1 \equiv \frac{1}{\sqrt{2}}(\phi_4 - i\phi_5), \quad q_2 \equiv \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2), \quad \psi \equiv \psi^2. \tag{4.14}$$

The action with the new fields becomes

$$\begin{aligned} S_{\mathbf{I}}^E = \frac{k}{4\pi^2 r}\int_{\mathrm{R}\times\mathrm{CP}^2} & d^5x\sqrt{|g|}\ \mathrm{tr}\bigg[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\epsilon^{\mu\nu\lambda\rho\sigma}J_{\mu\nu}\left(A_\lambda\partial_\rho A_\sigma - \frac{2i}{3}A_\lambda A_\rho A_\sigma\right) \\ & +\frac{1}{2}D_\mu\phi_3 D^\mu\phi_3 + |D_\mu q^A|^2 + \frac{2}{r^2}(\phi_3)^2 + \frac{4}{r^2}|q^2|^2 + \frac{13}{r^2}|q^1|^2 \\ & +|[\phi_3, q^A]|^2 + \frac{1}{2}|[q^A, \bar{q}_A]|^2 + \frac{1}{2}(\sigma^I)^A{}_B(\sigma^I)^C{}_D[q^B, \bar{q}_A][q^D, \bar{q}_C] - \frac{6}{r}\phi_3[q^2, \bar{q}_2] \\ & -\frac{i}{2}\chi^\dagger\gamma^\mu D_\mu\chi - i\psi\gamma^\mu D_\mu\psi - \frac{1}{8r}\chi^\dagger J_{\mu\nu}\gamma^{\mu\nu}\chi - \frac{1}{4r}\psi^\dagger J_{\mu\nu}\gamma^{\mu\nu}\psi + \frac{3i}{4r}\chi^\dagger\sigma^3\chi + \frac{3i}{2r}\psi^\dagger\psi \\ & -\frac{i}{2}\chi^\dagger[\phi_3, \chi] + i\psi^\dagger[\phi_3, \psi] + \sqrt{2}i\psi^\dagger[\chi_A, q^A] - \sqrt{2}i[\bar{q}_A, \chi^\dagger]\psi\bigg], \end{aligned} \tag{4.15}$$

where $\sigma^{I=1,2,3}$ are the Pauli matrices.

Before performing the path integral, let us first fix the gauge following [22, 23]. We choose the Coulomb gauge $D^m A_m = 0$ and impose the residual gauge fixing condition as

$\frac{d}{d\tau}\alpha = 0$ where $\alpha \equiv \frac{1}{\omega_{CP^2}}\int_{CP^2} A_\tau$ is the s-wave component (or holonomy) of A_τ. The holonomy α is the only zero mode of the quadratic action. The residual gauge fixing introduces the Haar measure to the path integral. Thus the index at large k becomes the integral of the 1-loop determinant by the holonomy α

$$I = \frac{1}{N!}\int \prod_{i=1}^{N}\left[\frac{d\alpha_i}{2\pi}\right]\prod_{i<j}^{N}\left[2\sin\left(\frac{\alpha_i-\alpha_j}{2}\right)\right]^2 \times I_{1-\text{loop}}. \tag{4.16}$$

To obtain the 1-loop determinant, we will use the various CP^2 harmonics carrying electric charges R_2. Some of them are constructed in [15, 32]. Let us first focus on the scalars in the hypermultiplet. The scalars have the following quadratic terms

$$\bar{q}_1\left[-D_\tau^2 - D^m D_m + \frac{13}{r^2}\right]q^1 + \bar{q}_2\left[-D_\tau^2 - D^m D_m + \frac{4}{r^2}\right]q^2, \tag{4.17}$$

where the time derivative is

$$D_\tau = \partial_\tau - i[\alpha,\ \ ] - \frac{\beta}{\beta r}R_1 - \frac{i\gamma_1}{\beta r}(j_1-j_2) - \frac{i\gamma_2}{\beta r}(j_2-j_3). \tag{4.18}$$

We need to use the charged SU(3) harmonics $Y^{l+3R_2,l}$ if $R_2 > 0$ or $Y^{l,l+3|R_2|}$ if $R_2 < 0$ according to R_2 charges of the scalar fields. Here, the charged harmonics Y^{l_1,l_2} carries R_2 charge $\frac{l_1-l_2}{3}$. Then the corresponding harmonics are $Y^{l,l+3}$ for q^1 and $Y^{l,l}$ for q^2 respectively, and they diagonalize the quadratic equation. The 1-loop determinant of the hyper scalars becomes

$$\begin{aligned}\det{}_{H,b} = &\prod_{\alpha\in root}\prod_{l=0}^{\infty}\prod_{m_1,m_2\in(l,l+3)}\sin\left(\frac{\alpha+m_i\gamma_i+i(2l+5)\beta}{2}\right)\sin\left(\frac{\alpha+m_i\gamma_i-i(2l+5)\beta}{2}\right)\\ &\times\prod_{\alpha\in root}\prod_{l=0}^{\infty}\prod_{m_1,m_2\in(l,l)}\sin\left(\frac{\alpha+m_i\gamma_i+i(2l+1)\beta}{2}\right)\sin\left(\frac{\alpha+m_i\gamma_i-i(2l+3)\beta}{2}\right).\end{aligned} \tag{4.19}$$

where $m_i\gamma_i = m_1\gamma_1 + m_2\gamma_2$ and m_i denote the two Cartan charges of (l_1, l_2) representation of SU(3) isometry.

For the complex fermion ψ, we introduce the four spinor basis on CP^2

$$\Psi_1 = Y^{l,l+3}\epsilon_+\,,\quad \Psi_2 = \gamma^\tau\gamma^m D_m Y^{l,l+3}\epsilon_+\,,\quad \Psi_3 = Y^{l,l}\epsilon_-\,,\quad \Psi_4 = \gamma^\tau\gamma^m D_m Y^{l,l}\epsilon_- \tag{4.20}$$

where Y^{l_1,l_2} is the charged SU(3) harmonics defined above. These four basis can diagonalize the fermion quadratic action

$$\psi^\dagger\left[-i\gamma^\tau D_\tau - iD^m\gamma_m - \frac{1}{4r}J_{mn}\gamma^{mn} + \frac{3i}{2r}\right]\psi\,. \tag{4.21}$$

One then obtains the 1-loop determinant for the fermion field in the hypermultiplet

$$\begin{aligned}\det{}_{H,f} = &\prod_{\alpha\in root}\prod_{l=0}^{\infty}\prod_{m_1,m_2\in(l,l+3)}\sin\left(\frac{\alpha+m_i\gamma_i+i(2l+5)\beta}{2}\right)\sin\left(\frac{\alpha+m_i\gamma_i-i(2l+5)\beta}{2}\right)\\ &\times\sin\left(\frac{\alpha-3i\beta}{2}\right)\prod_{\alpha\in root}\prod_{l=1}^{\infty}\prod_{m_1,m_2\in(l,l)}\sin\left(\frac{\alpha+m_i\gamma_i+i(2l+1)\beta}{2}\right)\sin\left(\frac{\alpha+m_i\gamma_i-i(2l+3)\beta}{2}\right).\end{aligned} \tag{4.22}$$

The first line corresponds to the 1-loop determinant from Ψ_1, Ψ_2 and the second line is from Ψ_3, Ψ_4. Combining the complex scalar and the fermion contributions, the final 1-loop determinant of the hypermultiplet is given by

$$\frac{\det_{H,f}}{\det_{H,b}} = \prod_{\alpha\in root}\frac{1}{\sin\left(\frac{\alpha+i\beta}{2}\right)} = x^{\varepsilon_0}\exp\left[\sum_{n=1}^{\infty}\sum_{i,j}^{N}\frac{1}{n}x^n e^{ni\alpha_{ij}}\right]. \tag{4.23}$$

where $\varepsilon_0 = \frac{1}{2}N^2$ is the Casimir energy for the hypermultiplet.

Let us move on to the vectormultiplet contribution. It is straightforward to compute the fermionic contribution by using the same spinor basis (4.20). The quadratic equation for χ^1 is given by

$$(\chi^1)^\dagger\left[-i\gamma^\tau D_\tau - i\gamma^m D_m - \frac{1}{4r}J_{mn}\gamma^{mn} + \frac{3i}{2r}\right]\chi^1. \tag{4.24}$$

The corresponding 1-loop determinant becomes

$$\begin{aligned}\det{}_{V,f} = &\prod_{\alpha\in root}\prod_{l=0}^{\infty}\prod_{m_1,m_2\in(l,l+3)}\sin\left(\frac{\alpha+m_i\gamma_i+i(2l+6)\beta}{2}\right)\sin\left(\frac{\alpha+m_i\gamma_i-i(2l+4)\beta}{2}\right)\\ &\times\sin\left(\frac{\alpha-2i\beta}{2}\right)\prod_{\alpha\in root}\prod_{l=1}^{\infty}\prod_{m_1,m_2\in(l,l)}\sin\left(\frac{\alpha+m_i\gamma_i+i(2l+2)\beta}{2}\right)\sin\left(\frac{\alpha+m_i\gamma_i-i(2l+2)\beta}{2}\right).\end{aligned} \tag{4.25}$$

The first line is again the 1-loop determinant of Ψ_1, Ψ_2 and the second line is from Ψ_3, Ψ_4.

The quadratic action of the vector field is

$$\begin{aligned}\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + i\epsilon^{\mu\nu\lambda\rho\sigma}A_\mu\partial_\nu A_\lambda J_{\rho\sigma} = &(D_m A_\tau)^2 + 2A_\tau\partial_\tau D_m A^m\\ &-A_m\left(D_\tau^2\delta^m_n + D^2\delta^m_n - D^m D_n - 6\right)A^n\\ &+4iA_\tau D_m A_n J^{mn} - 2iA_m D_\tau A_n J^{mn}.\end{aligned} \tag{4.26}$$

We find that the following vector harmonics form the complete basis of the 5 vector components

$$\mathcal{A}_\tau = Y^{l,l}, \quad \mathcal{A}^1_m = D_m Y^{l,l}, \quad \mathcal{A}^2_m = J_{mn}D^n Y^{l,l}, \quad \mathcal{A}^3_m = \epsilon_-^\dagger\gamma_m\gamma^n D_n Y^{l,l+3}\epsilon_+. \tag{4.27}$$

Here, $\mathcal{A}_\tau, \mathcal{A}^1, \mathcal{A}^2$ are real vectors and $\mathcal{A}^3$ is a complex vector. As we have already taken into account the zero mode of A_τ, which gives the holonomy α and Haar measure of the

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gauge group, the range of the harmonics $Y^{l,l}$ is therefore $l > 0$. Under the Coulomb gauge $D^m A_m = 0$, we can turn off the modes corresponding to $\mathcal{A}_m^1$. The other two real vectors $\mathcal{A}_\tau, \mathcal{A}_m^2$ mix each other in the quadratic action. Taking into account the determinant factors from the gauge fixing procedure, we obtain the 1-loop determinant for the real vectors

$$\prod_{\alpha\in root}\prod_{l=1}^{\infty}\prod_{m_1,m_2\in(l,l)}\left[\sin\left(\frac{\alpha+m_i\gamma_i+i(2l+2)\beta}{2}\right)\sin\left(\frac{\alpha+m_i\gamma_i-i(2l+2)\beta}{2}\right)\right]^{\frac{1}{2}}. \tag{4.28}$$

The complex vector $\mathcal{A}^3$ is an eigenvector of the quadratic equation (4.26) and its 1-loop determinant is

$$\prod_{\alpha\in root}\prod_{l=0}^{\infty}\prod_{m_1,m_2\in(l,l+3)}\sin\left(\frac{\alpha+m_i\gamma_i+i(2l+6)\beta}{2}\right)\sin\left(\frac{\alpha+m_i\gamma_i-i(2l+4)\beta}{2}\right). \tag{4.29}$$

We then collect the fermion and the vector contributions as well as the contribution from a scalar field ϕ^3. After the huge cancellation between the fermionic and bosonic contributions, we finally find that the 1-loop determinant of the vector multiplet is trivial

$$\frac{\det_{V,f}}{\det_{V,b}} = 1. \tag{4.30}$$

Combining the contributions from the vector and the hypermultiplet, we obtain the following superconformal index at infinite k

$$\begin{aligned} I(x,y_1,y_2)_{k\to\infty} &= \frac{x^{\varepsilon_0}}{N!}\int\prod_{i=1}^{N}\left[\frac{d\alpha_i}{2\pi}\right]\prod_{i<j}^{N}\left[2\sin\left(\frac{\alpha_i-\alpha_j}{2}\right)\right]^2\exp\left[\sum_{n=1}^{\infty}\sum_{i,j}\frac{1}{n}x^n e^{ni\alpha_{ij}}\right] \\ &= x^{\varepsilon_0}\prod_{m=1}^{N}\frac{1}{1-x^m}. \end{aligned} \tag{4.31}$$

It follows that the index receives the contributions from the states formed by a single letter $\phi_1 - i\phi_2$. This result agrees with the 6d superconformal index at infinite k and, therefore, agrees with the half-BPS index (4.4). We believe that the full superconformal index at finite k can be calculated by including the instanton contribution.

5 Supergravity

Let us briefly consider the $\mathrm{AdS}_7 \times \mathrm{S}^4$ geometry corresponding to the 6d (2,0) theory [11]. In case we need the complete AdS_7 geometry with S^5 boundary. The maximally supersymmetric $\mathrm{AdS}_7 \times \mathrm{S}^4$ geometry is

$$\begin{aligned} &ds^2 = R^2(-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho ds_{S^5}^2) + \frac{1}{4}R^2 ds_{S^4}^2, \\ &F_4 \sim N\epsilon_4, \qquad R/\ell_p = 2(\pi N)^{1/3}. \end{aligned} \tag{5.1}$$

The 5d unit sphere and 4d unit sphere are modded by

$$\frac{\mathrm{S}^5\times\mathrm{S}^4}{Z_k}. \tag{5.2}$$

The metrics on S^5 and S^4 are, respectively,

$$ds^2_{S^5} = ds^2_{CP^2} + (dy' + V)^2,$$
$$ds^2_{S^4} = d\vartheta^2 + \sin^2\vartheta d\chi'^2 + \cos^2\vartheta ds_{S^2}. \tag{5.3}$$

where χ' is the phase corresponding to the phase of $\phi_4 + i\phi_5$ and $dV = 2J$ is the Kähler 2-form on CP^2. The Z_k modding for the first and second cases are

$$(\mathbf{I})\ (y', \chi') \sim (y', \chi') + \frac{2\pi}{k}(1, 3),$$
$$(\mathbf{II})\ (y', \chi') \sim (y', \chi') + \frac{2\pi}{k}(1, -1). \tag{5.4}$$

Let us focus on the first case with the change of coordinates to

$$y' = \frac{y}{k},\ \chi' = \chi + \frac{3y}{k}, \tag{5.5}$$

with $y \in [0, 2\pi]$ and $\chi \in [0, 2\pi]$. The geometry becomes

$$\begin{aligned} ds^2 &= R^2\left[-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho ds^2_{CP^2} + \frac{1}{k^2}\sinh^2\rho(dy + kV)^2\right] \\ &\quad + \frac{R^2}{4}\left[d\vartheta^2 + \sin^2\vartheta\left(d\chi + \frac{3dy}{k}\right)^2 + \cos^2\vartheta ds^2_{S^2}\right], \\ F &\sim N\left(\mathcal{V}_{S^4} + \frac{3}{k}\sin\vartheta\cos^2\vartheta\ d\vartheta \wedge dy \wedge \mathcal{V}_{S^2}\right), \end{aligned} \tag{5.6}$$

where

$$\mathcal{V}_{S^4} = \sin\vartheta\cos^2\vartheta\ d\vartheta \wedge d\chi \wedge \mathcal{V}_{S^2}. \tag{5.7}$$

where $\mathcal{V}_{S^2}$ is the volume form of a unit 2-sphere.

The corresponding Type IIA geometry can be obtained by the relation:

$$ds^2_{11} = e^{-2\sigma/3}ds^2_{10} + e^{4\sigma/3}(dy + \mathcal{A})^2,$$
$$F^4_{11} = e^{4\sigma/3}F^4_{10} + e^{\sigma/3}F^3_{10} \wedge dy. \tag{5.8}$$

Some of NS-NS fields of σ, g_{MN}, B_{MN} and R-R fields $C_\mu, C_{\mu\nu\rho}$ are nonvanishing as $C_M dx^M = \mathcal{A}$ and $e^{\sigma/3}F^3_{10} = e^{\sigma/3}dB = \frac{3N}{k}\sin\vartheta\cos^2\vartheta d\vartheta \wedge \mathcal{V}_{S^2}$. The metric (5.6) containing $(dy + kV)^2$ and $(d\chi + 3dy/k)^2$ becomes

$$\frac{R^2}{4k^2}(4\sinh^2\rho + 9\sin^2\vartheta)(dy + \mathcal{A})^2 + \frac{R^2\sinh^2\rho\sin^2\vartheta}{4\sinh^2\rho + 9\sin^2\vartheta}(d\chi - 3V)^2, \tag{5.9}$$

where

$$\mathcal{A} = k\frac{4\sinh^2\rho\ V + 3\sin^2\vartheta d\chi}{4\sinh^2\rho + 9\sin^2\vartheta}. \tag{5.10}$$

Thus the relation (5.8) implies

$$e^{4\sigma/3} = \frac{R^2}{4k^2}(4\sinh^2\rho + 9\sin^2\vartheta), \tag{5.11}$$

and

$$\begin{aligned} e^{-2\sigma/3}ds_{10}^2 =& + R^2[-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho ds_{CP^2}^2] \\ &+ \frac{R^2}{4}(d\vartheta^2 + \cos^2\vartheta ds_{S^2}^2) + \frac{R^2\sinh^2\rho\sin^2\vartheta}{4\sinh^2\rho + 9\sin^2\vartheta}(d\chi - 3V)^2. \end{aligned} \tag{5.12}$$

The field strength are

$$\begin{aligned} F_{10}^4 &= Ne^{-4\sigma/3}\mathcal{V}_S^4, \\ F_{10}^3 &= \frac{3N}{k}e^{-\sigma/3}\sin\vartheta d\vartheta \wedge \mathcal{V}_{S^2}. \end{aligned} \tag{5.13}$$

Note that F_{10}^4 is for the D4 branes and F_{10}^3 is for the D6 branes.

The radius of the circle fiber y is of order

$$e^{2\sigma/3} \sim \frac{N^{1/3}}{k}\sinh\rho. \tag{5.14}$$

As we divide the AdS_7 space, we do not have a small compact circle and so it is hard to say the theory has been reduced to the type IIA theory. However the above radius says that the M-theory description is valid for $1 \leq k \lesssim N^{1/3}$. Since the dilation field diverges at the boundary, the ultraviolet physics at the boundary is the 6d physics. From the metric (5.12) in the string frame of type IIA, one can see that the curvature scale of the type IIA theory is of order $\sqrt{R^3/2k} \sim \sqrt{N/k}$ which is large when 't Hooft coupling $\lambda = N/k$ is large.

6 Conclusion and discussion

We have found the supersymmetric Yang-Mills Chern-Simons theories on $\mathrm{R} \times \mathrm{CP}^2$ which arise from the Z_k modding of the 6d(2,0) theory on $\mathrm{R} \times \mathrm{S}^5$ with additional twistings along the R symmetry direction. Depending on the twisting, the number of supersymmetries can be 4 or 12. Here for simplicity we have focused the analysis for 4 supersymmetric case with the supersymmetric spinor parameter which is a singlet under the SU(3) isometry of CP^2. The fluctuation analysis shows that the fields have the right conformal dimension as expected from the 6d consideration. Supergravity analysis shows that there are M-theory region and type IIA region and weakly coupled region even though the boundary between first two regions is not that distinct.

We have argued that the number of supersymmetries get enhanced for $k = 1, 2, 3$ cases when the nonperturbative effects are included. As there is a discrete coupling constant, there might be a chance that our theories are finite in UV and represent the 6d theory completely once the nonperturbative effects are included. If it is not finite, one may need higher derivative terms to capture the 6d physics properly. However, the presence of the

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Chern-Simons like term may not allow such a higher derivative term. A further exploration along this direction is necessary.

Our theories are good stepping stones for calculating the index function of the 6d (2,0) theory and we hope to report the result in near future. There seems to be several interesting ideas to pursue from the current point. There may be many BPS objects in the theory which is not apparent in first glance. The N^3 degrees of freedom on the 6d (2,0) theory [30] have been studied from various points of view [15, 18, 31] and our theory may provide a further evidence.

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A Convention for metrics and gamma matrices

The space-time metric has the mostly positive signature. The metric tensors on CP^2 and S^5 are, respectively,

$$\begin{aligned} ds^2_{\mathrm{CP}^2} &= d\rho^2 + \frac{\tau_3^2}{4}\sin^2\rho\cos^2\rho \; + \frac{\tau_1^2+\tau_2^2}{4}\sin^2\rho, \\ ds^2_{\mathrm{S}^5} &= ds^2_{\mathrm{CP}^2} + (dy+V)^2, \qquad\qquad V = \frac{\tau_3}{2}\sin^2\rho, \end{aligned} \tag{A.1}$$

where y is the U(1) fiber direction. The left-invariant SU(2) 1-forms are

$$\begin{aligned} \tau_1 &= -\sin\psi d\theta + \cos\psi\sin\theta d\varphi, \\ \tau_2 &= +\cos\psi d\theta + \sin\psi\sin\theta d\varphi, \\ \tau_3 &= +d\psi + \cos\theta d\varphi, \end{aligned} \tag{A.2}$$

such that $d\tau_i = \frac{1}{2}\epsilon_{ijk}\tau_j\wedge\tau_k$. The range of variables are $\rho\in[0,\frac{\pi}{2}], \theta\in[0,\pi], \varphi\in[0,2\pi], \psi\in[0,4\pi]$ and $y\in[0,2\pi]$. The volumes of CP^2 and S^5 are $\pi^2/2$ and π^3, respectively.

The vierbein $e^m = e_p^m dx^p$ for CP^2 is

$$e^1 = d\rho,\ e^2 = \frac{\tau_3}{2}\sin\rho\cos\rho,\ e^3 = \frac{\tau_1}{2}\sin\rho,\ e^4 = \frac{\tau_2}{2}\sin\rho. \tag{A.3}$$

Their inverse $e_m = e_m^p\partial_p$ is

$$e_1 = \partial_\rho,\ e_2 = \frac{2\tilde{\tau}_3}{\sin\rho\cos\rho},\ e_3 = \frac{2\tilde{\tau}_1}{\sin\rho},\ e_4 = \frac{2\tilde{\tau}_2}{\sin\rho}, \tag{A.4}$$

where

$$\tilde{\tau}_1 = -\sin\psi\partial_\theta + \frac{\cos\psi}{\sin\theta}(\partial_\varphi - \cos\theta\partial_\psi),$$
$$\tilde{\tau}_2 = +\cos\psi\partial_\theta + \frac{\sin\psi}{\sin\theta}(\partial_\varphi - \cos\theta\partial_\psi),$$
$$\tilde{\tau}_3 = +\partial_\psi. \tag{A.5}$$

The Kähler 2-form on CP^2 is

$$J = \frac{1}{2}J_{mn}e^m \wedge e^n = \frac{1}{2}dV = e^1 \wedge e^2 + e^3 \wedge e^4. \tag{A.6}$$

The spin-connection for the CP^2 is

$$w^{12} = -\frac{\tau_3}{2}\cos 2\rho,\ w^{34} = +\frac{\tau_3}{2}(1+\sin^2\rho),$$
$$w^{23} = w^{41} = +\frac{\tau_2}{2}\cos\rho,\ w^{31} = w^{42} = +\frac{\tau_1}{2}\cos\rho. \tag{A.7}$$

The vierbein on S^5 is

$$E^m = e^m\ (m = 1,2,3,4),\quad E^5 = dy + V_p dx^p. \tag{A.8}$$

The inverse vierbein on S^5 is

$$E_m = e_m - e_m^p V_p \partial_y\ (m = 1,2,3,4),\quad E_5 = \partial_y. \tag{A.9}$$

The spin connection for S^5 is

$$W^{mn} = w^{mn} - J^{mn}E^5,\ W^5{}_m = J_{mn}e^n. \tag{A.10}$$

Our notation for the Minkowski space-time gamma matrices for 6d and 5d is as follows:

$$(5d)\ \ \gamma^0 = 1_2 \otimes i\sigma_2,\ \gamma^{1,2,3} = \sigma_{1,2,3}\otimes\sigma_1,\ \gamma^4 = 1_2\otimes\sigma_3,\ \gamma^{01234} = i1_4,$$
$$(6d)\ \ \Gamma^\mu = \gamma^\mu\otimes\sigma_1\ (\mu = 0,1,\cdots 4),\ \Gamma^5 = 1_4\otimes\sigma_2,\ \Gamma^7 = \Gamma^{01\cdots 5} = -1_4\otimes\sigma_3. \tag{A.11}$$

The 6d spinor field λ and the supersymmetric parameter ϵ have the opposite chirality so that $\Gamma^7\lambda = \lambda, \Gamma^7\epsilon = -\epsilon$. With $B = i\sigma_2\otimes\sigma_1$, we get $B\gamma^\mu B^{-1} = -\gamma^{\mu^*} = -\gamma_\mu^T$. The spinors transform as **4** of $\mathrm{Sp}(2)_R = \mathrm{SO}(5)$ symmetry and the 5d Euclidean gamma matrices on **4** are

$$\rho_{1,2,3} = \sigma_{1,2,3}\otimes\sigma_3,\ \rho_4 = 1_2\otimes\sigma_2,\ \rho_5 = 1_2\otimes\sigma_1. \tag{A.12}$$

Our choice of Cartan for $\mathrm{Sp}(2)_R$ is $R_2 \sim \frac{1}{2}\rho_{45}$ and $R_1 \sim \frac{1}{2}\rho_{12}$ to fermionic fields. The charge conjugation operator acting on **4** of $\mathrm{Sp}(2)_R$ is $C = i\sigma_2\otimes\sigma_1$ such that $C\rho_I C^{-1} = \rho_I^T$. With $\hat{B} = B\otimes\sigma_3$, we get $\hat{B}\Gamma^M\hat{B}^{-1} = \Gamma^{M*} = \Gamma_M^T$. We require the the reality conditions on the spinors to be

$$\lambda = -\hat{B}C\lambda^*,\ \epsilon = \hat{B}C\epsilon^* \quad\Longrightarrow\quad \lambda = BC\lambda^*,\ \epsilon = BC\epsilon^* \tag{A.13}$$

on four component spinors.

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B Killing spinors

The Killing spinors [15, 32] on $R \times S^5$ are defined as follows:

$$\hat{\nabla}_M \epsilon_\pm = \frac{i}{2}\Gamma_M \tilde{\epsilon}_\pm = \pm\frac{i}{2}\Gamma_M \Gamma_0 \epsilon_\pm, \tag{B.1}$$

and $\epsilon_\pm = \hat{B}C\epsilon^*_\mp$ and $\tilde{\epsilon}_\pm = \pm\Gamma_0\epsilon$. Here we will be loose about 8 and 4 component spinors as the chirality condition $\Gamma^7\lambda = \lambda, \Gamma^7\epsilon = -\epsilon, \Gamma^7\tilde{\epsilon} = \tilde{\epsilon}$ leaves no ambiguity. The covariant derivative to the spinor on S^5 given as

$$\hat{\nabla}_M \epsilon = \left(\partial_M + \frac{1}{4}W_M^{AB}\Gamma_{AB}\right)\epsilon. \tag{B.2}$$

The covariant derivative on spinors in S^5 can be expressed in terms of that on CP^2 plus the derivative along the circle fiber.

$$\begin{aligned}
\hat{\nabla}_0 \epsilon &\equiv \partial_t \epsilon = \frac{i}{2}\gamma_0 \tilde{\epsilon}, \\
\hat{\nabla}_m \epsilon &\equiv \left[\nabla_m - V_m\partial_y + \frac{1}{2}J_{mn}\Gamma^{n5}\right]\epsilon = \left[\nabla_m - V_m\partial_y + \frac{i}{2}J_{mn}\gamma^n\right]\epsilon = \frac{i}{2}\gamma_m\tilde{\epsilon}, \\
\hat{\nabla}_5 \epsilon &\equiv \left[\partial_y - \frac{1}{4}J_{mn}\Gamma^{mn}\right]\epsilon = \left[\partial_y - \frac{1}{4}J_{mn}\gamma^{mn}\right]\epsilon = \frac{1}{2}\tilde{\epsilon},
\end{aligned} \tag{B.3}$$

where $m = 1,2,3,4$ and $V = V_m e^m = V_\mu dx^\mu, J = \frac{1}{2}J_{mn}e^m \wedge e^n$ and $\nabla_m = e_m^\mu \nabla_\mu$ is the spinor covariant derivative on CP^2. The Killing spinor equation is solved with

$$\tilde{\epsilon} = \pm\Gamma_0\epsilon = \pm\gamma_0\epsilon \tag{B.4}$$

The covariant derivative on the gaugino field is

$$\begin{aligned}
\hat{\nabla}_0 \lambda &\equiv \partial_t \lambda, \\
\hat{\nabla}_m \lambda &\equiv \left[\nabla_m - V_m\partial_y + \frac{1}{2}J_{mn}\Gamma^{n5}\right]\lambda = \left[\nabla_m - V_m\partial_y - \frac{i}{2}J_{mn}\gamma^n\right]\lambda, \\
\hat{\nabla}_5 \lambda &\equiv \left[\partial_y - \frac{1}{4}J_{mn}\Gamma^{mn}\right]\lambda = \left[\partial_y - \frac{1}{4}J_{mn}\gamma^{mn}\right]\lambda.
\end{aligned} \tag{B.5}$$

Let us split the spinors to eigenspinors $\gamma_{12}\epsilon^{s_1 s_2} = is_1\epsilon^{s_1 s_1}, \gamma_{34}\epsilon^{s_1\sigma_2} = is_2\epsilon^{s_1\sigma_2}$. Note that $\gamma^0\epsilon^{s_1 s_1} = i\gamma^{1234}\epsilon^{s_1 s_2} = -is_1 s_2\epsilon^{s_1 s_2}$. One solution of the Killing spinor is

$$\textbf{(I)} \quad \epsilon_+ \sim e^{-\frac{i}{2}t + \frac{3i}{2}y}\epsilon_0^{++} \tag{B.6}$$

with a constant spinor ϵ_0^{++}. It is singlet under the SU(3) isometry of CP^2. The more complicated three Killing spinors are nontrivial linear combinations of three spinors

$$\textbf{(II)} \quad \epsilon_+ \sim e^{-\frac{i}{2}t - \frac{i}{2}y}(e_1^{+-}, \epsilon_1^{-+}, \epsilon_1^{--}), \tag{B.7}$$

where ϵ_1 depends on CP^2 coordinates nontrivially. They form a triplet under the SU(3) isometry of CP^2. The detail expression is known but not important here.

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